A Model of Endogenous Extreme Events

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Abstract

Our paper addresses the question ”What determines susceptibility to extremes and duration of extreme episodes?” Extreme events and fragility play a large role in recent socioeconomic life, and in economic models, such as Allen and Gale (1998) and Gabaix (2008). It is therefore important to understand how the likelihood of extremes depends on behavior of economic agents. We model endogeneity in extremes using the concept of congestion-induced frictions, as in transaction cost and public good literature. We construct an economy with financial congestion and positive network externalities from match surpluses. Our model delivers three main results. First, fragility depends on differences in marginal substitution, or income inequality, between borrowers and lenders. A basic empirical analysis demonstrates significant links between inequality and crises for OECD countries. Second, extreme episodes last until marginal substitution rates converge, or expected costs rise. Third, a government policy that taxes resource transfers may increase the likelihood of extreme events.

Keywords: Endogenous Extremes; Financial Congestion; Fragility; Income Inequality

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1 Introduction

‘Financial markets...can be quite fragile and subject to crises of confidence. Unfortunately, theory gives little guidance on the exact timing or duration of these crises...’
Reinhart and Rogoff (2009), page xil.

The need to understand market extreme events has become compelling. This paper develops a simple model of endogenous extremes, based on positive and negative externalities. We address two salient aspects of modern financial markets: dynamics and endogeneity in extremes. By dynamics, we refer to recurring episodes of 'surprise' extreme events. By endogeneity, we refer to the effect of economic agents on the likelihood of extremes. The economic costs of extreme events can be prohibitive, including widespread risk of default, and an impaired trading process because prices are uninformative. Extreme events also carry large social and psychological costs, such as risk of spillovers and increased Knightian uncertainty in an unstable economy. Such situations are described in several ways in existing research, including the words fragility and susceptibility to extremes, so we use both terms interchangeably.

Discussions of extreme economic events often model them as due to exogenous shocks (Allen and Gale (2000b); Barro (2006); Gabaix (2008)). But sometimes the likelihood of extreme events is affected by economic agents. We build on existing literature to explore a possible explanation for endogenous extremes, based on interaction of congestion and network externality effects. Extreme events have externality features, since they affect many individuals in the national or global economic system, even though often precipitated by a relatively small number of individuals. Externalities cause inefficiency of the price system, therefore society may not pay the appropriate price for the extremes that it generates.

Congestion as a Comprehensive Umbrella. A number of researchers have analyzed financial extremes and crises, resulting in a variety of approaches. Rational approaches discuss bubble expectations, agency costs, multiple equilibria, and fire-sale externalities. The financial frictions approach emphasizes that liquidity needs at the bank and individual levels have aggregate ramifications. Behavioral finance focuses on psychological biases and inefficiencies on the part of banks and individuals. And econophysics research posits that crashes are an emergent property of systems with

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1Endogenous extremes may arise due to negligence, bounded rationality, excess risk taking, or corruption. 
2See Harris (2003); Phelps (2007); Weitzman (2007); and Caballero and Krishnamurthy (2008).
3For rational bubbles, agency costs and multiple equilibria, see Blanchard (1979), Bernanke and Gertler (1989), and Benhabib and Farmer (1999). For fire-sale externalities, see Allen and Gale (2007). For financial frictions and liquidity, see Brunnermeier and Pedersen (2009), and Allen et al. (2011).
interacting components.\textsuperscript{4} Several of these frameworks are presented in Panel A of Table 1. These multifarious approaches are non-nested and often develop independently of each other. It is therefore difficult to evaluate models of extreme events or develop policy aimed at mitigating extremes, in a manner that is credible to economists from different persuasions.

What can be done to remedy this situation? A possible approach is just to work within one framework (e.g. behavioral or frictions) at a time, and build slowly to more general results. Another approach, which we find persuasive, is to look for common ground in all of the theories, that is, to answer the question “What, if anything, do the various theories of extreme events have in common?”

A Common Theme. At least one common element emerges: congestion in financial trades. All the above theories are based on the idea that extreme events arise when ‘too many’ individuals start doing the same activity. The activity is typically one of the following: withdrawing money from banks, selling assets that no longer seem valuable, refraining from buying assets, demanding margins be repaid, or holding on to liquid assets so that entrepreneurs cannot borrow to finance projects. However, if a small enough group of individuals did this action, it would not matter. There has to be a critical mass for the system to become congested, which then causes markets to fail or crash. Such congestion is consistent with actions that arise because of rational beliefs, behavioral biases or any of the other proposed theories of extremes.

This formulation clarifies that extreme events occur not just because agents act in a certain way, but also because of system capacity. Just as a large highway can handle more traffic than a small road, a developed financial market can handle more people selling assets, because of another, countervailing effect, network externalities: the more people involved in the market, the greater likelihood of finding counterparties. This latter perspective is not emphasized in the literature of extreme events and crises. Therefore, the congestion approach has two advantages: it is inclusive and respectful to major frameworks for modeling extreme events; and it provides a way of modeling the important tension between costs and benefits of trading resources in financial markets.

How can we model congestion in financial markets? There is a mature public finance framework on congestion externalities (Baumol and Oates (1988); Cornes and Sandler (1996)), on which we build. Consider an economy with $I$ agents, indexed by $i = 1, \ldots, I$. As in the public finance literature each agent $i$ buys two types of good, a private consumption good $y_i$ and a congestible public good $q_i$. If the aggregate amount of the congestible public good is $Q \equiv \sum_i q_i$, then the likelihood of congestion in the system is represented by a function $C(Q)$. In our paper $Q$ denotes resource transfers in

\textsuperscript{4}For behavioral approaches, see Froot et al. (1992); Benartzi and Thaler (1995); Odean (1999). For econophysics research, see Montier (2002) and Sornette (2004).
the financial system. It includes financial market activities that raise the likelihood of congestion, mainly borrowing funds and trading assets, and in recent history, trading in credit transfers.

**Relation to Existing Approaches.** The congestion function has an immediate interpretation in terms of existing theories. In a rational setting, $C(\cdot)$ represents the likelihood of experiencing high agency costs or ending up in an inefficient equilibrium. In a behavioral setting, $C(\cdot)$ captures the possibility that overconfidence or herd behavior is strong enough to generate excessive trading. And in the frictions literature, $C(\cdot)$ measures the likelihood of large, simultaneous liquidity needs in a significant portion of market participants. Our approach accounts for endogenous extreme events regardless of the source—illiquidity, behavioral factors, or rational causes.

### 1.1 Contributions of Our Paper

We contribute to the literature in several respects. First, we characterize the endogenous probability of extreme events, using a micro-founded approach. Specifically, we derive mathematical expressions to characterize the ‘signature’ of dynamic, endogenous extremes. This signature suggests a novel prediction that inequality and crises are related, which is upheld in a basic empirical application. Second, we account for both positive and negative externalities in financial markets, within the same model. Third, our model demonstrates the conditions where government intervention is and is not justified in the face of extreme events. More generally, our framework allows us to discuss an economic approach to extreme events, using the lens of public economics.

The remainder of the paper is organized in the following manner. The rest of Section 1 gives an overview of congestion and network effects in financial markets. Section 2 constructs a static model of endogeneity of $C$, using a congestible public good framework. Section 3 develops this model to characterize dynamic, endogenous extremes. Section 4 provides an empirical application, and Section 5 concludes. Proposition proofs and extensions are in the Appendix.

### 1.2 Congestion and Network Effects in Markets

As the recent financial crisis demonstrated, markets can become congested with frightening speed and intensity, which precipitates extreme events. In anticipation of Section 2, we use $C$ to denote the likelihood of extreme events or tail risk, which can affect both real and financial sectors. One approach to understanding tail risk (Shin (2009)) is to consider $C$ to depend on a reasonable economic variable $x$, such as excess borrowing: that is, the researcher assumes $C = C(x)$, based on empirical evidence or economic intuition. While appealing as a starting point, this approach runs the risk of
allowing the researcher freedom to choose a tractable but potentially inaccurate characterization of tail risk. As we show below (Propositions 1 and 3, Corollary 1), the optimal \( C(\cdot) \) that arises from a micro-founded approach has surprising properties and policy implications, which might not be evident based on limited data samples or a priori reasoning.

Our framework summarizes both negative frictions and positive external effects present in modern financial markets. This approach captures two relevant aspects of financial markets. First, during times of crisis, many investors wish to perform the same type of resource transfer, such as selling bad stocks or diversifying. This results in crowded trades, and what is individually rational becomes impossible for anyone to do (Pedersen (2009)). Second, and importantly for economic development, in normal times increased use of financial markets is welfare-enhancing, since it loosens intertemporal budget constraints and permits profitable projects to be undertaken (Rajan and Zingales (2003); Allen and Gale (2007)). It is only when markets have too large demands placed on them (excess trading, complex securities, etc.) that congestion effects dominate. We use the term financial congestion to denote such situations. As shown in Table 1’s Panel A, financial congestion springs from a variety of sources.

The network effect connotes benefits from establishing a long term match between buyers and sellers of resources. These benefits have been studied in various settings such as two-sided matching (Roth and Sotomayor (1990)), employment decisions (Boorman (1975); Petrongolo and Pissarides (2001)), banking (Allen and Babus (2009)), and buyer-seller exchange (Kranton and Minehart (2001); Corominas-Bosch (2004)). Network effects arise due to increased transfers in financial markets, in the following manner: larger amounts of trading per agent make financial markets thicker and more active, which raises the likelihood of agents finding desirable transactions. This effect can be motivated in several ways. One motivation is that buyers and sellers are not anonymous: they often need a relationship or link to exchange resources. Such a link “makes possible or adds value to a particular bilateral exchange” (Kranton and Minehart (2001), p.485.) A second motivation (Allen and Gale (2000b); Leitner (2005)) is that strong connections lower the risk of financial contagion since intermediaries can share losses. A third motivation comes from research on matching in markets, using both game theoretic approaches and the aggregate matching function. Game theoretic analyses emphasize that exchange works well in thick, uncongested markets. When markets are congested, they may unravel and lead to market failure (Roth and Xing (1994); Roth and Xing (1997); Niederle et al. (2008)). The aggregate matching approach summarizes various frictions of agents trying to meet and transact with each other, using a matching function (Mortensen and Pissarides (1999); Petrongolo and Pissarides (2001)). Similar to the matching function in labor markets, our

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5For overviews of networks in economics, see Katz and Shapiro (1985); Shapiro and Varian (1998); Rauch and Casella (2001); Jackson (2008); and Easley and Kleinberg (2010). For matching in game theory, see Roth and Sotomayor (1990). For aggregate matching models, see Mortensen and Pissarides (1999).
network function $N$ summarizes the complex interaction of resource buyers and sellers in financial markets.

### 1.3 Related Literature

Our research builds on existing work in three areas: extreme events, liquidity, and aggregate externalities. Regarding extreme events, Flood and Garber (1980) devise empirical tests of bubbles, which they apply to the German hyperinflation. Jansen and de Vries (1991) investigate extreme prices in S&P500 stocks, and document that the 1987 crash’s magnitude was not very exceptional. Montier (2002) discusses the notion that crashes and outliers are endogenous, due to a preponderance of sellers relative to buyers. Danielsson and Shin (2003) model a scenario where unanticipated coordination leads to an endogenous increase in risk. Bazerman and Watkins (2004) suggests that certain ‘surprise’ events in modern society are predictable, since there may exist sufficient information to know that these events are imminent. Poon et al. (2004) estimate that for G5 stock markets only in 13 of 84 cases is there evidence of asymptotic dependence. They argue that the probability of systemic risk may be over-estimated in financial literature. Barro (2006) builds an economy that incorporates rare disaster risk, modelled as a large drop in the economy’s wealth endowment. When this model is calibrated to the global economy, it can explain the equity premium and low risk free rate puzzles, and helps account for stock market volatility. Gabaix et al. (2006) develop a theoretical model where stock volatility is driven by large investors’ trading in illiquid markets. Weitzman (2007) discovers that when agents consider extremes in a Bayesian asset return model, there is a reversal of all major asset pricing puzzles. Gabaix (2008) and Gabaix (2010) generalize the Barro framework to account for dynamic probability of extreme events. Reinhart and Rogoff (2009) conclude that the biggest factors in crises are excessive debt and sudden shifts of confidence. The authors emphasize a “This time is different” mentality, where market participants downplay the likelihood of extreme events during the boom period preceding crises. Adrian and Brunnermeier (2010) construct a measure, CoVaR, that summarizes the conditional likelihood of an institution’s experiencing a tail event. They document statistically significant spillover risk across US institutions. These papers underscore the importance of accounting for extreme events in markets. Our paper extends this literature by analyzing endogenous causes of dynamic extreme events.

Regarding liquidity, Brunnermeier and Pedersen (2009) demonstrate that liquidity needs are self-reinforcing. During bad times, such needs can precipitate financial crises. Pedersen (2009) describes a situation where major market players all require liquidity, which ends up causing a shutdown in markets. Similar frictions can arise in markets where capital is deployed in a sluggish manner, and when there is a discrepancy between sophisticated and naive investors, see Stein (2009) and...
Duffie (2010). Wagner (2011) demonstrates that when investors face liquidation risk in multiple assets, they hedge by selecting heterogeneous portfolios with lower diversification benefits. These papers correctly emphasize the importance of transaction costs and liquidity in market outcomes. As shown in Table 1, extreme events may occur because of other factors such as excessive government debt, confidence shifts, and over-borrowing. As we discuss in Section 4, liquidity does not always empirically relate to stock market changes during extreme periods. Therefore, we expand the focus of the liquidity literature with respect to extreme events, by using a broader congestion approach that incorporates other determinants of crises.

Regarding aggregate externalities, Banerjee (1992) and Bikhchandani et al. (1992) model how agents coordinate and disregard private information. Such herd behavior affects the stock market, as examined by Froot et al. (1992). Allen and Gale (1998) construct a model of banking panics that are related to the business cycle. In this model depositors rationally fear low returns after observing that leading economic indicators suggest a downturn, and therefore optimally withdraw all funds from banks. The authors show that bank runs can be efficient, although this result does not hold when runs lower asset returns, nor in the presence of a stock market. In their Theorem 5 and Corollary 5.1, the authors formalize the inefficiency of bank runs, which motivates central bank intervention. A related literature on bubbles suggests extreme events can be caused by rational indeterminacy, behavioral factors, or new technology (Blanchard (1979), Abreu and Brunnermeier (2003), and Hong et al. (2008), respectively). Allen and Gale (2007) show that bubbles may be precipitated by incentive and limited liability issues, which reduce the costs of individual risk taking. Danielsson and Zigrand (2008) construct a model where asset prices are determined in the presence of systemic risk. The authors argue that regulation can reduce the likelihood of systemic risk, but carries costs, such as increased risk premia and volatility, and the possibility of non-market clearing. Shin (2009) demonstrates a wedge between individual risk and systemic risk, based on the tendency of agents to coordinate during extreme periods. Bianchi (2010), and Bianchi and Mendoza (2011) analyze dynamic equilibrium models where excess borrowing increases financial system fragility. The authors show that these negative externalities raise the likelihood of extreme events, and entail significant welfare costs. Morris and Shin (2011) develop a model where adverse selection is amplified across market participants, when agents cannot compute maximal expected losses. Our paper extends this literature by treating the endogeneity of tail risk, accounting for both negative and positive externalities.
2 A Static Model of Extremes

Extreme probabilities can be exogenous or endogenous. Exogenous extremes arrive from outside the economic system and are truly acts of nature, from the perspective of the domestic economy. Endogenous extremes, by contrast, are generated and amplified within the economic system, by agents’ activity and interaction. We focus on a canonical form of economic interaction, namely, transfer of resources. Research on resource transfers is summarized in Panel B of Table 1. A key feature of modern financial markets is that they enhance agents’ ability to transfer resources, which involves either trading commodities and assets or moving assets across time. This activity can aid or harm individuals not party to the transfer.

2.1 Background for the Model

We consider a stylized model where agents use financial markets to transfer idle resources to the present from the future, or from one resource-plenty agent to a resource-scarce agent. An agent who participates in resource transfers faces three effects. She enhances her own utility; she contributes to decreased performance of the system whenever she excessively coordinates with others’ trading strategies (negative congestion externality); and she helps improve functioning of the system by enlarging diversification, matching and trade possibilities for others (positive network externality). We build on extant public economics frameworks (Baumol and Oates (1988); Myles (1995); Cornes and Sandler (1996)) and use two probability density functions: the congestion function \( C(\cdot) \), which measures the probability of congested trades in financial markets, and the network function \( N(\cdot) \), which summarizes enhanced trading opportunities due to thicker markets. We denote the derivative of the congestion function as the marginal likelihood of extremes. Since the term fragility has also been used in similar contexts, we use the term fragility and marginal likelihood of extremes interchangeably.

The Environment. The economy \( E \) is defined as a collection of goods \((q, y)\), prices \( \pi \) and preferences \( U \), that is, \( E = (q, y, \pi; U(\cdot)) \). Specifically, \( U(\cdot) \) is a utility function, \( y \) is a consumption good, \( q \) represents resource transfers; and \( \pi \) represents average prices. The environment comprises \( I \geq 2 \) identical individuals of whom we consider one representative individual. Each agent \( i \) buys a consumption good \( y^i \) and also trades \( q^i \) units of her resources. The total amount of trading is \( \sum_{i=1}^{I} q^i \equiv Q \), which affects the likelihood of congestion or friction in financial markets. Thus \( Q \) is

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Resource transfers include such activities as borrowing funds, and trading assets. Resource transfers are essential functions of financial markets, see Goetzmann and Rouwenhorst (2005). For specific congestion and network functions, see Edwards (1990).
a public good that negatively affects utility, since it raises the likelihood of socially harmful extreme events. Each agent has an exogenous income $W_i$, and takes as given the prices of $y$ and $q$, namely $\pi_y$ and $\pi_Q$. We normalize the price of $y$ to $\pi_y = 1$.

**Assumptions and Definitions.** Agent’s preferences are represented by a neoclassical utility function $U$ as in Allen and Gale (2007). $U(\cdot)$ is therefore quasi-concave, increasing and continuously differentiable. $U$ depends on goods, congestion $C$ and network effects $N$, that is, $U = U(y, q, C(\cdot), N(\cdot))$. As in public economics settings, congestion and networks depend on aggregate usage $Q$, that is, $C = C(Q)$ and $N = N(Q)$. Hence the likelihood of congested trades depends on the level of resource transfers in the economy. Throughout, we restrict attention to interior optima for illustrative purposes. Derivatives are denoted with a subscript. For example, $U_y \equiv \frac{\partial U}{\partial y}$, $U_q \equiv \frac{\partial U}{\partial q}$, and $U_c \equiv \frac{\partial U}{\partial c}$. Similarly, $C_Q \equiv \frac{\partial C}{\partial Q}$, and $N_Q \equiv \frac{\partial N}{\partial Q}$.

**Assumption 1.** The derivatives of the congestion and network functions satisfy $N_Q \geq 0$ and $C_Q \geq 0$. Thus, an increase in total resource transfers weakly increases both the network effect and the likelihood of congestion.

**Assumption 2.** The derivatives of the utility function satisfy $U_N \geq 0$ and $U_C \leq 0$. Thus, an increase in market thickness weakly increases utility, while an increase in congestion weakly decreases utility.

**Assumption 3.** For congestion to occur, at least two agents must use the markets. Consider a two-person economy where agents transfer $q_1$ and $q_2$, and denote congestion as $C(q_1, q_2)$. Assumption 3 says that $C(0, \cdot) = C(\cdot, 0) = 0$. This rules out cases where an autocrat can shut down markets unilaterally.

**Extreme Events.** Most definitions of extremes involve the notions of reference points and thresholds. The reference point is the benchmark to which data are compared and is typically a moment or quantile such as the mean or median. The threshold captures how far away from the reference point is considered extreme. In Friedman and Laibson (1989) the benchmark is the mean and the threshold is two standard deviations. In extreme value theory (de Haan and Ferreira (2006)), the benchmark is the historical extremum and the threshold is any positive number. An approach that we use to fix ideas is that of Barro (2006), where extremeness is defined by a threshold $b$. We denote the reference level of prices as $\bar{x}$, which for simplicity we consider to be the sample average.

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8The Barro (2006) context is output, not asset prices, and $b$ is set to 10% of output.
**Definition 1:** An **extreme event** is a drop in the asset price $\pi$ of magnitude $b$ or more relative to the sample average $\bar{\pi}$. That is, an extreme event occurs at time $t$ if the following relationship holds:

$$|\bar{\pi} - \pi_t| \geq b. \quad (1)$$

Since historically most large asset price changes have been drops (Fama (1965); Friedman and Laibson (1989)), we focus on price declines, which obviates the need for absolute values in (1). Therefore an extreme event of interest occurs if $\bar{\pi} - \pi_t \geq b$, or

$$\pi_t \leq \bar{\pi} - b. \quad (2)$$

**Definition 2:** An **exogenous** extreme event occurs when a large fall in asset prices is unrelated to any economic variable $v \in E$.

**Definition 3:** An **endogenous** extreme event occurs when a large fall in asset prices is related to some economic variable $v \in E$.

**Financial Congestion.** In public finance, congestion refers to a situation where individuals coordinate excessively and therefore overuse a resource. Similarly, we use the term financial congestion to denote a situation where agents coordinate too much on one side of the market, resulting in excess asset supply. Specifically, consider a simple economy with 2 types of agents, with endowments of good $y$ and congestible asset $q$. Each agent $i$ decides to demand or supply a certain amount $s_i$ of asset $q$. The aggregate supply curve for asset $q$ is $S \equiv \sum_i s_i$. Analogous to the public finance literature, we define the capacity of the economy as the maximum level of aggregate net asset supply $\bar{S}$ the market can absorb without prices experiencing an extreme event:

$$\bar{S}: \text{for all } S \geq \bar{S}, \pi \leq \bar{\pi} - b \quad (3)$$

Thus, $\bar{S}$ represents the level of asset supply beyond which equation (2) holds. Graphically this is depicted in Figure 2, and leads us to the following definition.

**Definition 4:** A **financial congestion** occurs when investors as a whole wish to sell more than the capacity $\bar{S}$ of risky assets. From equations (2) and (3) above, such a large increase in asset supply causes an extreme event. Thus we define congestion and extreme events jointly. In the sequel, the congestion function $C$ measures net asset supply, how much investors coordinate in selling off risky assets.

This definition does not appeal to information asymmetry or other frictions. There is a simple link between congestion and the extreme events: when markets are ‘congested’ in terms of excess
supply of an asset, then asset prices experience a precipitous drop or extreme event. This approach is closely related to banking literature on asset fire sale externalities, see Allen and Gale (2007). A related type of congestion is described by Afonso (2011). Since endogenous extremes arise because of congestion in our model, the marginal likelihood of an endogenous extreme event, or fragility, is given by $C_Q$, the derivative of the density function.

### 2.2 Solution to the Model

The representative individual has a quasi-concave utility function $U_i(y^i, q^i, C(Q), N(Q))$, where derivatives satisfy $N_Q \geq 0$, and $U_N \geq 0$. Thus, an increase in resource transfers increases network quality, and utility increases with network quality. $C_Q \geq 0$, and $U_C \leq 0$. That is, the likelihood of congestion increases as resource transfers increase; and the agent’s utility falls if congestion becomes more prevalent.\(^9\) Throughout we will focus on the derivative $C_Q$, which we term fragility or the marginal likelihood of extreme events. The importance of this quantity stems from the fact that it captures the responsiveness of extreme probability to a change in resource transfers.

**Private Optimum:** The representative individual chooses her level of resource transfers, in a Nash-Cournot setting, where she takes $\hat{Q} = \sum_{j \neq i} q^j$ as exogenously given. Thus her problem is

$$\max_{(y^i, q^i)} U(y^i, q^i, C(\hat{Q} + q^i), N(\hat{Q} + q^i))$$

subject to

$$y^i + \pi_Q q^i = W^i.$$  

The first order conditions are $U^i_y - \lambda = 0$ and $U^i_y + U^i_C C_Q + U^i_N N_Q - \lambda \pi_Q = 0$, which combine to give

$$\pi_Q^p = \frac{U^i_q + U^i_N N_Q}{U^i_y} + \frac{U^i_C C_Q^p}{U^i_y},$$

\hspace{1cm} (4)

or

$$C_Q^p = \frac{\pi_Q^p - \frac{U^i_q + U^i_N N_Q}{U^i_y}}{\frac{U^i_C C_Q^p}{U^i_y}},$$

\hspace{1cm} (5)

where the superscript $p$ denotes a private solution. $C_Q^p$ measures fragility, or the marginal likelihood of extreme events in a competitive economy.

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\(^9\)This formulation of congestion is standard in public economics, see Cornes and Sandler (1996).
Social Optimum: The corresponding social optimum solves the following program:

$$\max \sum_i U^i(y^i, q^i, C(Q), N(Q))$$

subject to

$$\sum_i y^i + \pi \sum q^i = \sum W^i.$$ 

The necessary first order conditions are

$$U^i_y = 0$$

and

$$U^i_q + \sum_j U^j_C = \lambda \pi,$$

which together yield

$$\pi^s_Q = \frac{U^i_q + \sum_j U^j_{N, NQ}}{U^i_y} + \sum_j \left( \frac{U^j_J}{U^j_y} \right) C^s_Q, \quad (6)$$

or

$$C^s_Q = \frac{\pi^s_Q - \sum_j \frac{U^j_{N, NQ}}{U^j_y} U^i_q}{\sum_j \left( \frac{U^j_{J}}{U^j_y} \right)}, \quad (7)$$

where the superscript $s$ denotes the social optimum. $C^s_Q$ is the marginal likelihood of extreme events in an economy where social costs are considered.

When do private markets yield inefficiently high levels of fragility? Consider a given price level $p = \pi Q$. Then comparison of conditions (5) and (7) reveals that the private likelihood of congestion $C^p = C^s_Q$ exceeds the social optimum $C^s_Q$ if the following inequality holds:

$$\pi^s_Q - \sum_j \frac{U^j_{N, NQ}}{U^j_y} U^i_q \sum_j \left( \frac{U^j_J}{U^j_y} \right) > \pi^s_Q + \sum_j \frac{U^j_{N, NQ}}{U^j_y} \sum_j \left( \frac{U^j_J}{U^j_y} \right).$$

Given the assumption of $I \geq 2$ identical individuals, superscripts can be removed to yield $U^i = U^j = U$, and $\sum_j U^j_C = IU_C$. Therefore the above inequality can be rewritten as

$$\frac{\pi U_y - U_N N Q}{U_C} > \frac{\pi Q U_y - U_q - IU_N N Q}{IU_C},$$

or

$$\frac{(I-1)[\pi Q U_y - U_q]}{IU_C} > 0.$$  

Note that the term $(I-1)/IU_C$ is negative since $U_C < 0$ at an interior optimum and $I > 1$. Therefore the inequality above implies $\pi Q U_y - U_q < 0$, or

$$\pi Q < \frac{U_q}{U_y}. \quad (8)$$
This expression depends on the relative size of network and congestion effects. To see this, return to equation (4), which in the identical individual case is

\[ \pi^P_Q = \frac{U_q}{U_y} + \frac{U_N N_Q + U_C C_Q}{U_y}. \]  

(9)

In light of expression (8), the second term on the right of (9) determines whether the private or social optimum dominates. Since \( U_N \geq 0 \) and \( U_C \leq 0 \) by Assumption 2, this term can be either positive or negative, depending on the relative size of congestion and network effects. If network effects \( U_N N_Q \) are relatively small, then (8) holds and fragility is higher in the private equilibrium. Economically speaking, in markets where benefits from network externalities are exhausted, individuals’ optimal behavior ends up raising the likelihood of extremes. Conversely, when network effects are large relative to the congestion effect, individuals left alone will generate only a small likelihood of extreme events. These observations lead us to the following Proposition.

**Proposition 1.** Consider a large economy \( E \) with both external congestion and network effects from resource transfers. In this economy, the equilibrium marginal likelihood of extreme events \( C^P_Q \) is smaller than the social optimum \( C^S_Q \) if and only if \( U_N N_Q > |U_C C_Q| \), that is, if marginal network benefits dominate congestion effects.

The likelihood of extreme events is dynamic to the extent that network effects \( U_N N_Q \) change over time. This raises the question of when network effects are likely to be largest. Intuitively, the marginal benefit from participation is larger when only few agents participate in markets. Perhaps more subtly, network effects can also arise in a market with high demand for borrowing, when financial innovation or education attracts additional agents with excess savings.

Can public policy err in targeting extreme events? The following Corollary demonstrates how a policy response in the face of endogenous extreme events can have unintended, adverse consequences.

**Corollary 1:** Consider a large economy \( E \) with both congestion and network effects from resource transfers. In this economy, if the network effect \( U_N N_Q \) dominates the congestion effect \( U_C C_Q \), a tax on financial congestion will raise the marginal likelihood of extreme events and tail risk.

Corollary 1 tells us that if the goal of public policy is to reduce endogenous extreme events, it must account for network effects or run the risk of precipitating further extreme events.\(^{10}\) It can be seen as an extension of results (Theorem 5 and Corollary 5.1) from Allen and Gale (1998), and Allen and Gale (2000b), to asset markets with negative and positive externalities.

\(^{10}\) Similar results are found in public economics, where the public good is over-supplied under private provision. See Buchanan and Kafoglis (1963), and Diamond and Mirrlees (1973).
3 Endogenous Extremes in a Dynamic Model

We now develop the model to explore the timing and duration of extremes. Consider an economy populated by two types of representative agents with differentiated resource or wealth endowments. Agents use the financial system to transfer resources between themselves for one period. Types 1 and 2 transfer resources to each other in the amounts $q_1$ and $q_2$, recognizing that these transfers might raise fragility. Agents know the congestion function, of the form derived in Section 2. They are not subject to irrational behavior or asymmetric information about the likelihood of extremes. Below, we characterize when fragility overshoots or undershoots the socially efficient level.

The economy lasts for two periods. The first period is $t$ and the second period is $t+1$, in order to distinguish subscripts that refer to time from those that refer to agents. The more agents engage in resource transfers such as excessive borrowing or investing in risky securities, the more likely it is that markets experience congestion. Thus, the probability $C_{t+1} = C_{t+1}(q_{1,t}, q_{2,t})$ of future extreme events increases with the average level of current resource transfers, $C_{i,t+1} = \frac{\partial C_{t+1}()}{\partial q_{i,t}} \geq 0$, $i = 1, 2$. Similarly, the probability $N_{i} = N_{i}(q_{1,t}, q_{2,t})$, where $N_{i} = \frac{\partial N_{i}()}{\partial q_{i,t}} \geq 0$, $i = 1, 2$. For concreteness, the two main agents each conduct only one type of resource transfer—selling or buying. We call these agents sellers and buyers, and use subscripts 1 and 2 to index variables pertaining to sellers and buyers, respectively.

Representative sellers and buyers have neoclassical utility functions $u_1$ and $u_2$, respectively, which depend on wealth: $u_i = u_i(w_i)$, where $u_i'(w_i) > 0$, $i = 1, 2$. To control for contemporaneous costs, we consider utility to be net of current costs. In the first period sellers and buyers interact and transfer resources, and with probability $N_{i} = N_{i}(q_{1,t}, q_{2,t})$, where $N_{i} = \frac{\partial N_{i}()}{\partial q_{i,t}} \geq 0$, $i = 1, 2$. For concreteness, the two main agents each conduct only one type of resource transfer—selling or buying. We call these agents sellers and buyers, and use subscripts 1 and 2 to index variables pertaining to sellers and buyers, respectively.

11 Similar assumptions occur in other economic contexts, such as price taking, competitive agents in Arrow and Debreu (1954) and Debreu (1959), even though the demand of each agent will affect price to some extent. Such myopic behavior can be found in other rational settings: investors with log utility decide their portfolios without reference to future investment opportunities, see Ingersoll (1987), chapter 11.

12 This indicates that excessive transfers are destabilizing, without emphasizing the channel of destabilization. Particular channels are in Panel B of Table 1.
In the second period $t+1$, the buyer and seller receive exogenous wealth endowments $\bar{w}_1$ and $\bar{w}_2$, respectively. Given local nonsatiation, budget constraints hold with equality. If an extreme event occurs in the future (second period), agent $i$ incurs a finite cost $k_{i,t+1} > 0$ net of interest, $i = 0, 1, 2$. Therefore the seller’s second period constraint, which accounts for the possibility of costly extreme events, is

$$w_{1,t+1} = \bar{w}_1 + C_{t+1}(\cdot)[q_{1,t}(1 + i) - k_{1,t+1}] + [1 - C_{t+1}(\cdot)][q_{1,t}(1 + i)],$$

which simplifies to $\bar{w}_1 + q_{1,t}(1 + i) - C_{t+1}(\cdot)k_{1,t+1}$. Similarly, the buyer’s second period constraint is $w_{2,t+1} = \bar{w}_2 - q_{2,t}(1 + i) - C_{t+1}(\cdot)k_{2,t+1}$.

### 3.1 The Signature of Extreme Events

Given an interest rate $i$ and discounting factor $\beta$, at period $t$ the seller decides how much resources $q_1$ to transfer by maximizing $u_1(w_{1,t}) + \beta u_1(w_{1,t+1})$ subject to the constraints from above:

$$w_{1,t} = -q_{1,t} + N_t(\cdot)s_{1,t},$$

$$w_{1,t+1} = \bar{w}_1 + q_{1,t}(1 + i) - C_{t+1}(\cdot)k_{1,t+1}.$$

After substituting the constraints into the utility arguments, first order conditions for an interior solution are $u'(w_{1,t})[-1 + N^1_{Q,t}(\cdot)s_{1,t}] + \beta u'(w_{1,t+1})[(1 + i) - C^1_{Q,t+1}(\cdot)k_{1,t+1}] = 0$, which can be rewritten as

$$C^1_{Q,t+1}(\cdot) = -\frac{u'(w_{1,t})[1 - N^1_{Q,t}(\cdot)s_{1,t}]}{\beta u'(w_{1,t+1})k_{1,t+1} + (1 + i)k_{1,t+1}}. \tag{10}$$

Equation (10) says that optimally fragility depends on the marginal rate of substitution for transferring resources between periods $t$ and $t + 1$, discounted by expected costs.

Similarly, the buyer chooses $q_2$ to maximize $u_2(w_{2,t}) + \beta u_2(w_{2,t+1})$, subject to

$$w_{2,t} = q_{2,t} + N_t(\cdot)s_{2,t},$$

$$w_{2,t+1} = \bar{w}_2 - q_{2,t}(1 + i) - C_{t+1}(\cdot)k_{2,t+1}.$$
After substituting constraints into the utility arguments, the first order conditions for an interior solution are \( u'_2(w_{2,t})[1 + N^2_{Q,t}(\cdot)s_{2,t}] - \beta u'_2(w_{2,t+1})[(1 + i) + C^2_{Q,t+1}(\cdot)k_{2,t+1}] = 0 \), which can be rewritten

\[
C^2_{Q,t+1}(\cdot) = \frac{u'(w_{2,t})[1 + N^2_{Q,t}(\cdot)s_{2,t}]}{\beta u'(w_{2,t+1})k_{2,t+1}} - \frac{(1 + i)}{k_{2,t+1}}. \tag{11}
\]

As in equation (10), the above expression implies that fragility depends on expected costs and the intertemporal marginal rate of substitution.

**Equilibrium:** In equilibrium, effective demand and supply of resource transfers will be equal\(^{14}\), \( q_1 = q_2 \equiv q \). For illustrative purposes, consider a symmetric equilibrium where buyers and sellers have identical utility functions, benefits and costs, \( u_1 = u_2 = u \), \( s_1 = s_2 = s \), and \( k_1 = k_2 = k \). Now equate optimality conditions for seller and buyer in (10) and (11) for a given level of network effect \( N \equiv N^1 = N^2 \). Thus

\[
- \frac{u'(w_{1,t})[1 - N_{Q,t}(\cdot)s_{t}]}{\beta u'(w_{1,t+1})k_{t+1}} + \frac{(1 + i)}{k_{t+1}} = \frac{u'(w_{2,t})[1 + N_{Q,t}(\cdot)s_{t}]}{\beta u'(w_{2,t+1})k_{t+1}} - \frac{(1 + i)}{k_{t+1}},
\]

which implies

\[
1 + i = \frac{1}{2\beta} \left[ \frac{u'(w_{1,t})[1 - N_{Q,t}(\cdot)s_{t}]}{u'(w_{1,t+1})} + \frac{u'(w_{2,t})[1 + N_{Q,t}(\cdot)s_{t}]}{u'(w_{2,t+1})} \right].
\]

Substituting this expression in (11) and simplifying, we obtain that equilibrium fragility satisfies

\[
C^p_{Q,t+1}(\cdot) = \frac{1}{2\beta k_{t+1}} \left[ \frac{u'(w_{2,t})[1 + N_{Q,t}(\cdot)s_{t}]}{u'(w_{2,t+1})} - \frac{u'(w_{1,t})[1 - N_{Q,t}(\cdot)s_{t}]}{u'(w_{1,t+1})} \right], \tag{12}
\]

where the superscript \( p \) denotes a private market outcome. Equation (12) constitutes the signature of endogenous extremes. The responsiveness of extreme probability to resource transfers is proportional to the differential in marginal rates of substitution, which are weighted by the network effect \( N_{Q,t}(\cdot)s_{t} \). When there is a big difference in marginal rates of substitutions, susceptibility to extreme events is higher. To the extent that marginal utility, expected costs, and network surpluses change over time, it follows that extreme probability is dynamic. If extremes were truly exogenous, there would be no statistical relation between extreme probability and \( q \).

There are three main effects in (12). First is the \( k_{t+1} \) cost term in the denominator: the larger it is, the less likelihood of congestion. This effect is intuitive because large expected costs reduce net utility, therefore inducing agents to transact less and lowering financial congestion. Second is the difference in marginal rates of substitution \( \frac{u'(w_{2,t})}{u'(w_{2,t+1})} - \frac{u'(w_{1,t})}{u'(w_{1,t+1})} \): the larger this is, the higher likelihood of extremes. This result is sensible because a larger wealth inequality makes it more attractive for individuals to participate in markets to transfer resources, which in turn increases the likelihood of congestion. Third and similarly, fragility increases with the network effect \( N_{Q,t}s_{t} \). This is intuitive because markets with larger matching opportunities attract participation in transferring resources.

\(^{14}\)The Appendix demonstrates a case where transfers might not be equal.
and hence raise the likelihood of financial congestion. We graphically depict these effects in Figures 3 and 4 below. We summarize the results from (12) in the following Proposition.

**Proposition 2.** In a dynamic economy with symmetric preferences, network effects, and external costs of extremes, fragility $C_Q^o$ is potentially dynamic. $C_Q^o$ decreases with expected costs of extreme events $k_{t+1}$; and increases with both the network surplus and with inequality between agents’ marginal rates of substitution.

An immediate result of the above Proposition is that fragility becomes arbitrarily low when network-weighted marginal rates of substitution are equated. This gives us information about timing and expected duration of extreme events, as summarized in the following Corollary.

**Corollary 2.** In a dynamic economy with symmetric preferences, network effects, and external costs of extremes, extreme episodes will persist until the network-weighted marginal rates of substitution are equated, or until expected costs of extremes rise high enough.

**Social Optimum.** Suppose a social planner forces the seller (type 1) to consider the effect of her selling on other agents, and therefore to internalize benefits $s_{2,t}$ and costs $k_{2,t+1}$. Her problem is similar to that preceding equation (10), except that the budget constraints become

$$w_{1,t} = -q_{1,t} + N_t(s_{1,t} + s_{2,t})$$

$$w_{1,t+1} = \bar{w}_1 + q_{1,t} \cdot (1 + i) - C_{t+1}(k_{1,t+1} + k_{2,t+1})].$$

Solving the first order conditions as before, we obtain the counterpart of equation (10):

$$C^1_{Q,t+1}(\cdot) = -\frac{u'(w_{1,t})[1 - 2N_{1,t}(\cdot)s_t]}{2\beta k_{t+1} \cdot u'(w_{1,t+1})} + \frac{1 + i}{2k_{t+1}},$$

where, as before, we let $s = s_1 = s_2$, $u = u_1 = u_2$, and $k_{t+1} = k_{1,t+1} = k_{2,t+1}$, and consider a fixed marginal network effect $N = N^1 = N^2$. Similarly, if the buyer accounts for society-wide external benefits and costs, her optimization yields

$$C^2_{Q,t+1}(\cdot) = \frac{u'(w_{2,t})[1 + 2N_{1,t}(\cdot)s_t]}{2\beta k_{t+1} \cdot u'(w_{2,t+1})} - \frac{1 + i}{2k_{t+1}}.$$

Equating the conditions (13) and (14), as before, gives

$$1 + i = \frac{1}{2\beta} \left[ \frac{u'(w_{1,t})[1 - 2N_{Q,t}(\cdot)s_t]}{u'(w_{1,t+1})} + \frac{u'(w_{2,t})[1 + 2N_{Q,t}(\cdot)s_t]}{u'(w_{2,t+1})} \right].$$
Substituting the above expression in (14), we obtain the socially optimal fragility:

\[ C_{Q,t+1}^s(\cdot) = \frac{1}{4\beta k_{t+1}} \left[ \frac{u'(w_{2,t})[1 + 2N_{Q,t}(\cdot)s_t]}{u'(w_{2,t+1})} - \frac{u'(w_{1,t})[1 - 2N_{Q,t}(\cdot)s_t]}{u'(w_{1,t+1})} \right] \]  

(15)

The quantities in equations (12) and (15) will differ in general. It is in this sense that competitive markets may lead to endogenous, inefficient probability of extreme events.\(^{15}\) We are not just saying there is a link between excessive resource transfers and extremes. Instead, we are showing that even without asymmetric information or irrationality, inefficient extreme events may arise as an equilibrium phenomenon. This phenomenon occurs due to the failure of both resource sellers and buyers to internalize important externalities, namely, the benefits of the network surplus from matching and the expected costs from financial congestion.

We require conditions to determine a priori whether the likelihood of extremes is larger in the competitive equilibrium or the social optimum. We summarize our results from comparing equations (12) and (15) in the following Proposition:

**Proposition 3.** In a dynamic economy with symmetric preferences, network effects and external costs of extremes, equilibrium fragility is in general not socially optimal. Equilibrium fragility \( C_{Q}^p \) exceeds the social optimum \( C_{Q}^s \) if and only if systemic costs exceed utility-weighted network benefits.

Proposition 3 characterizes equilibrium fragility as excessively large when the social costs from transferring resources dominate the social benefits. This is consistent with Proposition 2 above. In economic terms, private agents are willing to put up with a higher fragility than is socially optimal, since they do not bear all the net social costs. This result may be useful in suggesting micro-founded criteria for policymakers who wish to tackle systemic risk. For example, taxing financial transactions may have the unintended consequence of reducing network benefits.

**An Example with Power Utility.**

The signature of extremes in Proposition 2 can be specialized to a particular utility function, in order to apply equation (12) to real world data. We use power utility, of the form \( u(w) = \frac{w^{1-\gamma}}{1-\gamma} \), where \( \gamma \) is the coefficient of relative risk aversion. The derivative is \( u'(w) = w^{-\gamma} \), which we substitute into (12) to obtain

\[ C_{Q}^p = \frac{1}{2\beta k} \left[ \left( \frac{w_{2,t+1}}{w_{2,t}} \right)^\gamma [1 + N_{Q}s] - \left( \frac{w_{1,t+1}}{w_{1,t}} \right)^\gamma [1 - N_{Q}s] \right], \]  

(16)

\(^{15}\)Optimality will not necessarily entail complete elimination of extreme events. Rather, the extreme probability is adjusted to the point where marginal benefit to sellers of an additional unit of the externality-generating activity, \( u_1'(q) \), equals marginal cost to other agents, \(-u_2'(q)\).
where for emphasis we remove time subscripts on quantities that are indexed by only one time period. Equation (16) suggests that in a power utility setting, the likelihood of financial extreme events depends on inequality in income growth between buyers and sellers.

3.2 Summary and Implications

The preceding results have implications for regulatory policy and risk management. Proposition 2 cautions risk managers against the assumption that exposure to extreme events is constant over time. It also suggests possible warning signals for regulators and risk managers—low expected costs and a large gap between agents’ desires to transfer resources over time. More generally, Proposition 2 predicts that developments to enhance resource transfers (including financial innovation and loose interest rates) will affect the likelihood of extremes. Proposition 3 suggests a role for regulators to intervene and prevent excessive financial extremes. However, it advises regulators that intervention might inadvertently increase the likelihood of extremes, in a manner that has previously received little attention.\\footnote{16}

More tentatively, the results may also have relevance for current financial issues in the Euro zone. Proposition 2 suggests that the increased fragility was precipitated by low expected costs previously (e.g. on the part of Greek debtors) and by a large inequality between marginal rates of substitution for borrowers and lenders of capital. After expected costs rise or inequalities are ameliorated, then the elevated extremes will begin to play out.

Measures of congestion and network effects are already used in public economics, algorithmic game theory and network economics. In light of the above prominence of these quantities, it may be valuable for policymakers to create and utilize congestion and network indices for financial markets.

4 Empirical Application

Although this paper is primarily theoretical, the importance of some ideas may be illustrated in the following applications. We consider three issues. First, we assess the value of considering a comprehensive congestion model, based on study of recent extreme events. Second, we calibrate a

\footnote{16}{In principle, regulators could tax ‘excessive’ transfers. However, this requires extensive monitoring of investor positions. A more realistic approach involves combining regulation with enhanced education about costs of extreme events, and the role of individuals and institutions in precipitating these costs. This approach is similar to recent education about other externalities such as effects of drunk driving, cigarette smoking, and human impact on the natural environment.}
congestion model to US data in order uncover implications for fragility. Finally, we investigate the empirical link between income inequality and financial crises.

4.1 Value of a Comprehensive Approach

We examine four cases of well-known extreme events in the Dow Jones Industrial Average: the October 1987 stock market crash, the liquidity crisis of August 1998, Lehman Brothers’ bankruptcy in September 2008, and the flash crash of May 2010. We choose daily data for two variables that are recognized as drivers of crises, namely, liquidity and confidence. Liquidity is measured using turnover and spreads, while confidence is measured with VIX.¹⁷ In the preliminary analysis we find that the magnitude of changes for liquidity and VIX is much larger than in the DJIA itself, which is not satisfactory for theories that attempt to explain large stock price changes. We also examine rank correlations of the liquidity and confidence measures with the DJIA,¹⁸ with surprising results. Only the confidence measure VIX is always significantly correlated with the DJIA. In just one case, May 2010, is a liquidity measure (Turnover) correlated with the DJIA. The fact that liquidity measures are typically uncorrelated with the DJIA is not reassuring, since there is a robust body of theoretical research linking liquidity to extreme financial events.

Therefore, in a further effort to understand the relation of liquidity and confidence factors with the stock market, we extract the two largest orthogonal principal components from the liquidity measures and VIX around each period. We then calculate correlations of the principal components with liquidity, VIX and the DJIA. The results are presented in Table 2. Panel A presents standard correlation coefficients. In all periods the DJIA is significantly correlated with at least one principal component, and is correlated with both components for 1987 and 1998. The relation is somewhat complicated since the various liquidity measures never load on just one component. For example, in 1987 SPREAD is significantly correlated only with the largest principal component COMP1, while ALIQ and TURN (along with VIX) are significantly correlated with both principal components. In Panel B, we examine rank correlations, which are more robust. The DJIA is only correlated with one principal component in each period, not necessarily the largest one. As before, the liquidity and VIX measures display a complex relation, since some liquidity measures load on the same component as VIX while others do not. For example, in 1998 TURN (VIX) is only correlated with COMP1 (COMP2), while ALIQ and SPREAD are significantly correlated with both components. There are three broad lessons from Table 2: the DJIA is not always correlated with the largest principal component from the various candidates; in each period the liquidity measures never load exclusively

¹⁷VIX is the implied volatility of an at-the-money 30-day option, which reflects investor confidence.
¹⁸These results are available in the supplement.
on one factor; and the confidence and liquidity measures sometimes load on the same component and sometimes do not, thereby displaying a rather complex relationship. Thus, focusing on a single factor may not suffice to capture the forces behind market crashes. These results are in favor of a general model of financial congestion, which attributes crises to a combination of standard forces such as confidence and liquidity.

4.2 Calibration to US Data

We calibrate expression (16) to US data, and present results in Table 3. First consider Panel A, where costs of extremes represent 10% of endowment, as in Barro and Jin (2011). In the case of zero network effects or “pure congestion” (column two), fragility $C_p^Q$ is always less than 0.05. Thus, a unit increase in resource transfers raises the likelihood of extreme events by less than 5%. By contrast, the third column of Panel A shows that, for a very small network surplus of 0.01, fragility always exceeds 10%. Thus, when we account for network effects, equilibrium fragility is more than twice as large. When we consider larger network effects of 0.1 in column 5, the discrepancy is even greater. Panel B’s results assume a higher cost of extremes, equal to 15% of endowment, as in Barro (2006). A similar pattern emerges, namely, the maximal fragility of 0.03 (column two) under pure congestion is less than half the minimal fragility of 0.07 (column 3) when we consider even a small network effect.

We display fragility for small and large network effects in Figures 3 and 4, respectively. These figures corroborate the results in Proposition 2 and Table 3. Specifically, the fragility surface is the same shape in both figures: for all risk aversion levels, fragility increases with the network surplus. Moreover, fragility rises to higher levels in Figure 4 (large network effect) than in Figure 3 (small network effect). Thus, fragility is more prominent in a world of large surpluses from network matches. The above findings have important policy implications. In particular, by ignoring network effects, policymakers may under-estimate fragility. This result underscores the importance of accounting for both financial congestion and network effects, in stress tests and policy analysis.

4.3 The Link between Income Inequality and Crises

Proposition 2 suggests a novel implication, namely that crisis exposure depends on income inequality between buyers and sellers. Assessing this implication is difficult because it requires reliable data on both inequality and crises at the same frequency, ideally for a panel of countries. While we do not have micro data on buyers and sellers of assets, there are data on household inequality,
namely the Gini coefficient for disposable income, available from the database of Solt (2009). In order to proxy for crisis exposure, we use the BCDI measure of Reinhart and Rogoff (2009), which counts the total number of crises each year in a large panel of countries. While these data are by no means ideal, they may be useful for a basic exploration of the relation between inequality and crises in developed financial markets. Since there can be multiple determinants of crises, we attempt to control for omitted variable bias by using fixed effects.

We examine 26 OECD countries which are listed in Table 4. Countries are chosen that have continuous data available for both inequality and the crisis index over the largest period of overlap in the two databases. The results of a simple test of association are in Table 4. Countries are ranked into three equal groups based on average inequality over the sample. Then the average crisis index is computed for each group, and tested for equality. Interestingly, the average crisis index increases monotonically as we move from the lowest to the highest inequality levels. Moreover, the Kruskal-Wallis test says we can reject the null that the crisis index is the same across country portfolios, thus providing some evidence in favor of inequality-related heterogeneous exposure to crises.

The preceding association results say nothing about explanatory or predictive power of inequality. Therefore, we analyze regression models of the crisis index on current and lagged inequality. Since there are evidently many causes of crises, we control for omitted variable bias by incorporating fixed country effects. Table 5, Panel A presents results from a negative binomial regression model, a flexible model for count data, where the dependent variable is the crisis index. The explanatory variables are inequality (INEQ) and an interaction term of inequality the previous three years (INEQ123). The most striking finding is that, even in the presence of fixed effects, both current and lagged inequality have significant explanatory power for the number of crises. Panel B performs a similar analysis for changes in the crisis index, and finds that only average inequality of the sample (AVINEQ) is significant. Furthermore, the large p-values of the test for fixed effects versus the null hypothesis of random effects, indicate that random effects are dominant. Thus, to predict changes in crisis exposure for our sample, stable country covariates are not as important as average inequality over the years for each country.

In summary, our empirical application is cause for optimism, because the theoretical link between inequality and crisis exposure is reasonably borne out by the data. Although our focus in this paper is primarily theoretical, these empirical findings are highly suggestive, and worth exploring in followup research.

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19Only relevant parameter estimates are reported. Other results are available from the author upon request.
5 Conclusions

Our paper develops a simple approach to endogenous extreme events. We suggest that fragility may vary systematically over time, and be explained on the basis of financial congestion. Our results extend research by Allen and Gale (2000b) to asset markets with both negative and positive externalities. Moreover, we distinguish exogenous from endogenous extremes, the latter of which can be understood in a public good framework. This distinction has immediate policy implications: for truly exogenous extremes, we must focus on ex post protection, while for endogenous extremes, we can in principle use economic incentives to entice agents to reduce extremes themselves. We have three main contributions. First, we develop the ‘signature’ of endogenous extremes, and provide insight on their incidence: According to Proposition 2, markets are more susceptible to extremes if expected costs are low and there is a large discrepancy between marginal rates of substitutions for borrowers and lenders. In simple specifications this discrepancy translates into income inequality, and a basic empirical application finds a significant relationship between income inequality and crises across OECD countries. Second, extreme episodes last until marginal rates of substitution converge, or expected costs rise. Third and perhaps most interesting, Proposition 3 suggests limits on the role for central bank and regulatory intervention. In tackling issues related to economic instability, regulators can ameliorate fragility, but only if they account for countervailing network effects.

While our paper describes a method for understanding patterns in fragility, it does not aim to predict all possible extreme events. The aim is to show that, far from being random, the probability of various types of endogenous extremes may have similar patterns. We accomplish this aim using a broad congestion-network approach, which allows us to relate the likelihood of extremes to economic variables: resource transfers, surplus from buyer-seller matching, expected costs and marginal utility of income. Our framework may be seen as a step towards incorporating dynamic, endogenous, extremes into standard economic analysis. Acknowledgement of endogenous extremes may also be helpful for risk management. Important extensions include estimating financial congestion, identifying dynamic extremes in particular markets, and exploring various channels of endogenous extremes encountered in practice. Such refinements present an exciting task for future research.
References


A Proofs of Propositions

We present proofs of all propositions except Proposition 2 and Corollary 2, which follow directly from equation (12).

Proposition 1: Consider a large economy $E$ with both external congestion and network effects from resource transfers. In this economy, the equilibrium marginal likelihood of extreme events $C_p^Q$ is smaller than the social optimum $C_s^Q$ if and only if $U_N N_Q > |U_C C_Q|$, that is, if marginal network benefits dominate congestion effects.

Proof. We have to show that $C_p^Q < C_s^Q$ if and only if $U_N N_Q > |U_C C_Q|$. By comparing equilibrium and socially optimum congestion in Section 2, we obtained the necessary and sufficient condition for $C_p^Q > C_s^Q$ in (8), namely, $\pi_Q^p < U_q / U_y$. Reversing the sign, we have that $C_p^Q < C_s^Q$ under the following condition:

$$\pi_Q^p > U_q / U_y.$$ 

In order to assess implications of this inequality condition, rewrite the optimum as in (9):

$$\pi_Q^p = U_q / U_y + U_N N_Q + U_C C_Q / U_y.$$ 

The second term on the right determines whether the inequality condition holds. Since $U_N \geq 0$ and $U_C \leq 0$ by Assumption 2, the inequality condition obtains if $U_N N_Q > |U_C C_Q|$, as was to be shown. ■

Corollary 1: Consider a large economy $E$ with both congestion and network effects from resource transfers. In this economy, if the network effect $U_N N_Q$ dominates the congestion effect $U_C C_Q$, a tax on financial congestion will raise the likelihood of extreme events and tail risk.

Proof. Suppose that $U_N N_Q > |U_C C_Q|$, and consider that a policymaker imposes the optimal tax $t$. We have to show that imposing $t$ will raise the competitive likelihood of extremes $C_p^Q$ to a new level $C_{Q^{New}} > C_p^Q$.

By definition, the optimal tax $t$ is such that when agents pay the price $\hat{\pi} = \pi_Q^p + t$, their maximizing behavior leads to the socially optimum congestion level.\(^{20}\) Thus, the new likelihood of extremes is $C_{Q^{New}} = C_s^Q$. But by Proposition 1 above, if $U_N N_Q > |U_C C_Q|$, then $C_s^Q > C_p^Q$. Therefore $C_{Q^{New}} > C_p^Q$, as was to be shown. ■

Proposition 3: In a dynamic economy with symmetric preferences, network effects, and external costs of extremes, equilibrium fragility is in general not socially optimal. Equilibrium fragility $C_p^Q$ exceeds the social optimum $C_s^Q$ if and only if the systemic costs exceed utility-weighted network benefits.

\(^{20}\)For a derivation of the optimal tax in a congestion setting, see Cornes and Sandler (1996), pp. 275-276.
Proof: Regarding social non-optimality, this follows directly by comparing the equilibrium \( C^p_t \) and social optimum \( C^q_t \) in (12) and (15). Regarding the condition for \( C^p > C^q \), from the expressions in (12) and (15), and by positivity of the common coefficient \( \frac{1}{\beta w_{t+1}} \) under the maintained assumptions, \( C^p > C^q \) holds under the following condition:

\[
\frac{2u'(w_{2,t})[1 + N_{Q,t}(\cdot)s_t]}{u'(w_{2,t+1})} - \frac{2u'(w_{1,t})[1 - N_{Q,t}(\cdot)s_t]}{u'(w_{1,t+1})} > \frac{u'(w_{2,t})[1 + 2N_{Q,t}(\cdot)s_t]}{u'(w_{2,t+1})} - \frac{u'(w_{1,t})[1 - 2N_{Q,t}(\cdot)s_t]}{u'(w_{1,t+1})},
\]

which simplifies to

\[
\frac{u'(w_{2,t})}{u'(w_{2,t+1})} > \frac{u'(w_{1,t})}{u'(w_{1,t+1})}.
\]

(17)

We now use the above relationship in the optimal conditions in order to obtain the desired inequality. First, express the signature of extremes from equation (12) in terms of marginal rates of substitution as

\[
\frac{u'(w_{2,t})}{u'(w_{2,t+1})} = \frac{u'(w_{1,t})}{u'(w_{1,t+1})} \cdot \frac{1 - N_{Q,t}(\cdot)s_t}{1 + N_{Q,t}(\cdot)s_t} + \frac{2\beta C_{Q,t+1}(\cdot)k_{t+1}}{1 + N_{Q,t}(\cdot)s_t}.
\]

(18)

The only way for (17) to hold, i.e. \( \frac{u'(w_{2,t})}{u'(w_{2,t+1})} > \frac{u'(w_{1,t})}{u'(w_{1,t+1})} \), is if the quantity in (18) exceeds \( \frac{u'(w_{1,t})}{u'(w_{1,t+1})} \), or equivalently,

\[
\frac{u'(w_{1,t})}{u'(w_{1,t+1})} \left[ \frac{-2N_{Q,t}(\cdot)s_t}{1 + N_{Q,t}(\cdot)s_t} \right] + \frac{2\beta C_{Q,t+1}(\cdot)k_{t+1}}{1 + N_{Q,t}(\cdot)s_t} > 0.
\]

Since \( N_{Q,t}(\cdot) > 0 \) under the maintained assumptions, we ignore the denominator and this condition becomes

\[
-2N_{Q,t}(\cdot)s_t \frac{u'(w_{1,t})}{u'(w_{1,t+1})} + \beta C_{Q,t+1}(\cdot)k_{t+1} > 0,
\]

which simplifies further to

\[
\beta C_{Q,t+1}(\cdot)k_{t+1} > N_{Q,t}(\cdot)s_t \frac{u'(w_{1,t})}{u'(w_{1,t+1})}.
\]

(19)

The left side of (19) is the discounted marginal systemic cost, and the right side is the weighted (by marginal rate of substitution) network benefit, as was to be shown.

B Mixed Strategy Equilibria

In the main text we modelled agents as choosing one optimal level of transfers in a symmetric equilibrium, since our purpose was to highlight a source of endogeneity and discuss policy effectiveness. It is also instructive to consider other strategic equilibria. For example, faced with uncertainty about others’ behavior, buyers and sellers might play a mixed strategy, where they potentially choose different resource transfers. One interesting possibility is that sometimes they separate themselves from the herd and get out of the market \((q = 0)\), and sometimes they stay in \((q > 0)\). This is a simple way to represent uncertainty about whether one’s trading partners will leave one ‘holding the bag’. We formalize this possibility with a basic, illustrative, model of strategy. We consider a similar model to Section 3, except that there is an intermediate period, say, \( t + \frac{1}{2} \), when agents can
get out of the market before the extreme events. For simplicity, we ignore network effects $N_s$, sum the monetary payoffs in $t$ and $t + 1$, and remove time and other subscripts. Players are small enough that each individual does not affect the aggregate likelihood $C$. Period $t$ payoffs are always $q$ for the buyer and $-q$ for the seller, which we add to the period $t + 1$ payoffs below.

**Payoffs:** There are four scenarios to consider, representing entries in the Table below. For the top left entry, both players stay in the market, so their payoffs are as analyzed in the text, and they receive $w_{1,t+1} = w_1 + q(1 + i) - Ck$ (seller) and $w_{2,t+1} = w_2 - q(1 + i) - Ck$ (buyer). For the bottom right entry, both players get out of the market before period $t + 1$, and both receive 0. For the bottom left entry, if the seller plays $q = 0$, she avoids the $k$ costs of extreme events, but also foregoes the interest and pays a fee. The seller therefore receives only a fraction of her original resource transfer, $fq < q$. Thus the seller’s payoff in period $t + 1$ is now $w + fq$. The buyer repays $fq$, so her period $t + 1$ payoff is $w_1 - fq - Ck$. Similarly, for the top right entry, if the buyer gets out before period $t + 1$, she avoids the $k$ costs of extreme events, but must pay a proportional fee $f$ to the seller. So her period $t + 1$ payoff is now $w_2 - q(1 + i + f)$. The payoff matrix is presented below.

<table>
<thead>
<tr>
<th></th>
<th>Stay In ($q &gt; 0$)</th>
<th>Buyer</th>
<th>Get Out ($q = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seller</td>
<td>$-q + w_1 + q(1 + i) - Ck$, $q + w_2 - q(1 + i) - Ck$</td>
<td>$-q + w_1 + q(1 + i + f) - Ck, q + w_2 - q(1 + i + f)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-q + w_1 + fq, q + w_2 - Ck$</td>
<td>$0, 0$</td>
<td></td>
</tr>
</tbody>
</table>

**Mixed Strategies.** Suppose the buyer chooses a distribution over her actions—she stays in ($q > 0$) with probability $\delta$ and gets out ($q = 0$) with probability $1 - \delta$. For the seller to respond optimally with a mixed strategy, the seller must be indifferent between staying in and getting out of the market. That is,

$$\delta[-q + w_1 + q(1 + i) - Ck] + (1 - \delta)[-q + w_1 + q(1 + i + f) - Ck] = \delta[-q + w_1 + fq] + (1 - \delta)[0],$$

which simplifies to

$$\delta = \frac{Ck - w_1 - q(i + f)}{q(1 - 2f) - w_1}. \quad (20)$$

Similarly, if the seller stays in (gets out) with probability $\gamma (1 - \gamma)$, then the buyer must be indifferent between staying in and getting out, for this to be part of a Nash equilibrium. That is,

$$\gamma[q + w_2 - q(1 + i) - Ck] + (1 - \gamma)[q + w_2 - fq - Ck] = \gamma[q + w_2 - q(1 + i + f)] + (1 - \gamma)[0],$$

or

$$\gamma = \frac{Ck - q(1 - f) - w_2}{q(2f - 1) - w_2}. \quad (21)$$

**Calibrating Mixed Strategy Probabilities.** In order to calibrate $\delta$ and $\gamma$, we assign plausible values for the parameters. We allow the probability of extremes $C$ to vary between 0 and 1, since it
is endogenous in our model. Similarly, resource transfers \( q \) are expressed as a fraction of endowment \( \bar{w} \) between 0 and 1, \( q = q\bar{w} \). As in Barro and Jin (2011), we set the costs of extreme events at 10% of wealth, \( k = 0.1\bar{w} \). We assume that the fees \( f \) for getting out of the market are similar in expected terms to the cost of extreme events, otherwise everyone would get out of the market and it would collapse—therefore \( f = k = 0.1w \). The interest rate \( i \) is set equal to 0.071, which is calculated by the author as the average of the monthly prime bank loan rate from January 1949 to December 2008 using data from the Federal Reserve Bank of Saint Louis. Using these calibrations, the mixed strategies for the buyer and seller satisfy:

\[
\delta = \frac{0.1C\bar{w}_1 - \bar{w}_1 - q\bar{w}_1(0.71 + 0.1)}{\bar{w}_1[q(1 - 0.2) - 1]} = \frac{1 + 0.81q - 0.1C}{1 - 0.8q}
\]

\[
\gamma = \frac{0.1C\bar{w}_2 - q\bar{w}_2(1 - 0.1) - \bar{w}_2}{qw_2(0.2 - 1) - \bar{w}_2} = \frac{1 + 0.9q - 0.1C}{1 + 0.8q}
\]

These numbers show the implied mixed strategies for buyers and sellers, and are represented in Figure 1. Evidently, the seller’s \( \gamma \) is a valid probability, between 0 and 1, for much of its range. However, the buyer’s \( \delta \) is valid for a more restricted range of \( q \).

**Figure 1: Mixed Strategies**

The figure represents the mixed strategy in equations (20) and (21), using the calibration described above. \( \delta \) and \( \gamma \) are the probabilities of staying in the market for the buyer and seller, respectively. \( C \) denotes the likelihood of extremes. \( q \) denotes resource transfers.
Table 1: Different Faces of Congestion and Resource Transfers

Panel A: Different Aspects of Financial Congestion

<table>
<thead>
<tr>
<th>Form of Congestion</th>
<th>Researchers</th>
<th>Main Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess borrowing</td>
<td>Fisher (1933); Reinhart and Rogoff (2009)</td>
<td>Preponderance of debt ⇒ economy susceptible to shocks</td>
</tr>
<tr>
<td>Lack of liquidity</td>
<td>Allen and Gale (2007); Pedersen (2009); Brunnermeier and Pedersen (2009)</td>
<td>Simultaneous needs for liquidity ⇒ fragile credit markets</td>
</tr>
<tr>
<td>Over-confidence</td>
<td>Odean (1999); Barber and Odean (2000)</td>
<td>Excessive trading ⇒ market inefficiency</td>
</tr>
</tbody>
</table>

Panel B: Different Types of Resource Transfers

<table>
<thead>
<tr>
<th>Form of Resource Transfer</th>
<th>Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>Over-borrowing, leverage</td>
<td>Fisher (1933); Minsky (1982)</td>
</tr>
<tr>
<td>Excessive credit creation</td>
<td>Allen and Gale (2000a)</td>
</tr>
<tr>
<td>Large sales due to confidence shifts</td>
<td>Reinhart and Rogoff (2009); Keynes (1936)</td>
</tr>
<tr>
<td>Increased Trading</td>
<td>Montier (2002); Gabaix et al. (2006)</td>
</tr>
<tr>
<td>Fire-sale asset liquidation</td>
<td>Allen and Gale (2007)</td>
</tr>
</tbody>
</table>

Figure 2: Extreme Price Changes and Asset Supply Externalities

The figure shows the relation between a shift in aggregate supply from \( S \) to supply capacity \( \bar{S} \), and an extreme event in prices. The drop in prices from \( \pi \) to \( \pi_1 \) is an extreme event as defined in expression (2) because it reaches the threshold \( b \).
The table presents Pearson and Spearman (rank) correlations of the Dow Jones Industrial Average with the largest principal components from liquidity and confidence measures. COMP1 and COMP2 denote the largest and second largest principal components, respectively. ALIQ denotes the market average of the Amihud (2002) liquidity measure. SPREAD and TURN denote the average dollar bid-ask spread and turnover, respectively. VIX is the implied volatility of an at-the-money 30-day option, which reflects investor confidence. All variables are in percentage changes relative to the previous trading day. The time period for each extreme event comprises the four months surrounding the month of the event. For the October 1987 crash, the sample period is August 1, 1987 to December 31, 1987. For the August 1998 LTCM event, the sample period is June 1, 1998 to October 31, 1998. For the September 2008 episode around the Lehman Brothers Bankruptcy, the sample period is July 1, 2008 to November 30, 2008. For the May 2010 flash crash the period is March 1, 2010 to July 31, 2010. P-values are in parentheses.

### Panel A: Pearson Correlations

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>COMP1</strong></td>
<td><strong>COMP2</strong></td>
<td><strong>COMP1</strong></td>
<td><strong>COMP2</strong></td>
</tr>
<tr>
<td><strong>COMP1</strong></td>
<td>-0.4637</td>
<td>0.2082</td>
<td>-0.0628</td>
</tr>
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<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
</tr>
<tr>
<td><strong>COMP2</strong></td>
<td>-0.5124</td>
<td>-0.7363</td>
<td>-0.8157</td>
</tr>
<tr>
<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
<td>(0.5223)</td>
<td>(&lt;.0001)</td>
</tr>
<tr>
<td><strong>ALIQ</strong></td>
<td>-0.2294</td>
<td>-0.6643</td>
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<tr>
<td>(0.0180)</td>
<td>(&lt;.0001)</td>
<td>(0.0752)</td>
<td>(&lt;.0001)</td>
</tr>
<tr>
<td><strong>TURN</strong></td>
<td>0.8482</td>
<td>0.8036</td>
<td>0.7678</td>
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<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
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<tr>
<td><strong>SPREAD</strong></td>
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<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
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<tr>
<td><strong>VIX</strong></td>
<td>0.5526</td>
<td>-0.1376</td>
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<tr>
<td>(&lt;.0001)</td>
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### Panel B: Spearman (Rank) Correlations

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>COMP1</strong></td>
<td><strong>COMP2</strong></td>
<td><strong>COMP1</strong></td>
<td><strong>COMP2</strong></td>
</tr>
<tr>
<td><strong>COMP1</strong></td>
<td>-0.0103</td>
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<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
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<td><strong>COMP2</strong></td>
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<td>(&lt;.0001)</td>
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<td><strong>ALIQ</strong></td>
<td>-0.5156</td>
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<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
</tr>
<tr>
<td><strong>TURN</strong></td>
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<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
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<tr>
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</tr>
<tr>
<td>(&lt;.0001)</td>
<td>(0.0522)</td>
<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
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<tr>
<td><strong>VIX</strong></td>
<td>0.2258</td>
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<td>0.1633</td>
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<tr>
<td>(0.0199)</td>
<td>(0.0015)</td>
<td>(&lt;.0001)</td>
<td>(&lt;.0001)</td>
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</tbody>
</table>
The table presents the calibrated financial fragility, from Equation (12). We use a power utility specification as in (16). $N_{Qs}$ is the network surplus from matching and transacting in markets, as a fraction of wealth. The quantities needed to calibrate this equation include risk aversion ($\gamma$), discount factor $\beta$; expected costs $k$; wealth changes for asset buyers $\frac{w_{2,t+1}}{w_{2,t}}$; and wealth changes for asset sellers $\frac{w_{1,t+1}}{w_{1,t}}$. We choose $\gamma$ between 1 and 10, and $\beta = 0.99$ as in Mehra and Prescott (2003).

Panel A sets $k = 0.1$ as in Barro and Jin (2011) and Panel B sets $k = 0.15$ as in Barro (2006). To calculate $\frac{w_{2,t+1}}{w_{2,t}}$ and $\frac{w_{1,t+1}}{w_{1,t}}$, we use average quarterly changes from 1985 to 2011 for total household assets and total bank assets in the US, yielding $\frac{w_{2,t+1}}{w_{2,t}} = 1.0149$ and $\frac{w_{1,t+1}}{w_{1,t}} = 1.0141$, respectively. These data are obtained from the FRED database at the Federal Reserve Bank of St. Louis.

<table>
<thead>
<tr>
<th>Panel A: Extreme Costs $k = 0.1$</th>
<th>$N_{Qs} = 0$</th>
<th>$N_{Qs} = 0.01$</th>
<th>$N_{Qs} = 0.1$</th>
<th>$N_{Qs} = 0.5$</th>
<th>$N_{Qs} = 1$</th>
<th>$N_{Qs} = 2$</th>
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<table>
<thead>
<tr>
<th>Panel B: Extreme Costs $k = 0.15$</th>
<th>$N_{Qs} = 0$</th>
<th>$N_{Qs} = 0.01$</th>
<th>$N_{Qs} = 0.1$</th>
<th>$N_{Qs} = 0.5$</th>
<th>$N_{Qs} = 1$</th>
<th>$N_{Qs} = 2$</th>
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<td>$\gamma$</td>
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<td>7.8074</td>
<td>15.5841</td>
</tr>
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</table>
Figure 3: Fragility Surface: Small Network Surplus

The figure calibrates the fragility surface, computed from equation (12). We use a power utility formulation and the calibration values from Table 3. Fragility $C^p_Q$ denotes the increase in likelihood of extremes stemming from an increase in resource transfers. The network surplus $N_Qs$ represents the external benefit from matching and transacting in markets, as a fraction of wealth. $k$ is the fraction of wealth lost during an extreme event. Further examples of data values displayed in the figure are given explicitly in Table 3.

Figure 4: Fragility Surface: Large Network Surplus

The figure calibrates the fragility surface, computed from equation (12). We use the power utility formulation and calibration values from Table 3. Fragility $C^p_Q$ denotes the increase in likelihood of extremes stemming from an increase in resource transfers. The network surplus $N_Qs$ represents the external benefit from matching and transacting in markets, as a fraction of wealth. $k$ is the fraction of wealth lost during an extreme event. Further examples of data values displayed in the figure are given explicitly in Table 3.
Table 4: Tests of Association between Income Inequality and Crisis

The table presents results of nonparametric tests of association between crisis and inequality. The crisis index is the BCDI measure of Reinhart and Rogoff (2009), computed for the following 26 OECD countries: Australia, Austria, Belgium, Canada, Chile, Denmark, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Japan, Korea, Mexico, Netherlands, Norway, Poland, Portugal, Spain, Sweden, Switzerland, Turkey, the UK, and the USA. Countries are chosen for which there are data every year for both the crisis index and the inequality index. The inequality index is a Gini coefficient of household disposable income, obtained from the database of Solt (2009). Countries are ranked into three equal groups (terciles) based on average inequality over the sample. Then the average crisis index is computed for each tercile, and tested for equality using the Kruskal-Wallis test statistic, denoted $\chi^2$. The total number of observations is 754. The data are of annual frequency, and the sample period is from 1981 to 2009.

<table>
<thead>
<tr>
<th>Average Crisis Index for Inequality-based Country Portfolios</th>
<th>Mean</th>
<th>S. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Group 1:</strong> Low Inequality</td>
<td>Inequality Index 24.9585 (2.7678)</td>
<td>Crisis Index 0.5096 (0.7104)</td>
</tr>
<tr>
<td><strong>Group 2:</strong> Medium Inequality</td>
<td>Inequality Index 29.8072 (2.1558)</td>
<td>Crisis Index 0.6897 (0.9061)</td>
</tr>
<tr>
<td><strong>Group 3:</strong> High Inequality</td>
<td>Inequality Index 37.7857 (7.2278)</td>
<td>Crisis Index 1.0153 (1.1766)</td>
</tr>
<tr>
<td><strong>Test for Equality of Crisis Indices</strong></td>
<td>$\chi^2$</td>
<td>p-value</td>
</tr>
<tr>
<td></td>
<td>25.1819</td>
<td>&lt; .0001</td>
</tr>
</tbody>
</table>
Table 5: Fixed Effects Models of the Crisis Index

The table presents estimation results of the relation between crisis and inequality. The crisis measure is the BCDI index of Reinhart and Rogoff (2009), computed annually for the following 26 OECD countries: Australia, Austria, Belgium, Canada, Chile, Denmark, Finland, France, Germany, Greece, Hungary, Ireland, Italy, Japan, Korea, Mexico, Netherlands, Norway, Poland, Portugal, Spain, Sweden, Switzerland, Turkey, the UK, and the USA. Countries are chosen for which there are data every year for the BCDI index and the inequality index. The inequality index is a Gini coefficient of household disposable income, obtained from the database of Solt (2009). In all cases, we take the natural log of inequality. Panel A presents results from a negative binomial regression model. In Panel A the dependent variable is the crisis index. The explanatory variables are inequality (INEQ), an interaction term of inequality the previous three years (INEQ123), and fixed country effects. Estimation is done by maximum likelihood, with a scaled covariance matrix to correct overdispersion. The total number of observations is 676. χ denotes the Wald chi-square test statistic, and LR denotes the likelihood ratio test statistic for goodness of fit without INEQ and INEQ123. Panel B presents results from a cumulative logit model. The dependent variable is the change in the crisis index relative to the previous year. The explanatory variables are average inequality over the full sample (AVINEQ), the difference between average inequality and inequality from the previous year (DIFINEQ1), the difference between average inequality and inequality two years prior (DIFINEQ2), and fixed year effects. DIFINEQ1 and DIFINEQ2 represent fixed country effects because they depend only on variation over time within countries (Cameron and Trivedi (2005)). Estimation is done by robust White (1980) generalized estimation methods. Z and χ denote statistics for significance of parameters and significance of fixed effects versus the null of random country effects, respectively. The total number of observations is 702. Data are of annual frequency, and the sample period is 1981 to 2009.

### Panel A: Negative Binomial Model for Number of Crises

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>χ</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-10.0681</td>
<td>9.01</td>
<td>0.0027</td>
</tr>
<tr>
<td>INEQ</td>
<td>3.8555</td>
<td>9.87</td>
<td>0.0017</td>
</tr>
<tr>
<td>INEQ123</td>
<td>-0.0877</td>
<td>6.21</td>
<td>0.0127</td>
</tr>
</tbody>
</table>

Test of Overall Significance

<table>
<thead>
<tr>
<th>LR</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>77.0698</td>
<td>&lt; 0.0001</td>
</tr>
</tbody>
</table>

### Panel B: Cumulative Logit Model for Changes in Crisis Index

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Z</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIFINEQ1</td>
<td>4.6555</td>
<td>1.42</td>
<td>0.1562</td>
</tr>
<tr>
<td>DIFINEQ2</td>
<td>-4.1439</td>
<td>-1.35</td>
<td>0.1773</td>
</tr>
<tr>
<td>AVINEQ</td>
<td>0.6290</td>
<td>3.74</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Fixed vs Random Effects

<table>
<thead>
<tr>
<th>χ</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.7</td>
<td>0.2591</td>
</tr>
</tbody>
</table>