Financial Implications of Extreme and Rare Events

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Abstract

The recent financial crisis has reinforced the critical need for economic policy to anticipate and reduce the impact of unexpected, extreme, events. In this paper, we develop a framework based on latent regime shifts to analyze why investors may undertake overly risky investment strategies and how policymakers may attempt to constrain such behavior. This framework explains, in particular, why banks, investors, and policymakers may decide not to hedge against extreme events, even when those events are exogenously determined and have well understood probabilities and consequences. We also examine cases in which the extreme events are endogenously created by the investment strategies themselves. Our most striking finding is that the private costs of sustained suboptimal investment may be bounded and small. Thus, investors may knowingly ignore or exacerbate the likelihood of extreme events, especially if they face information costs to learning the structure of the financial environment in which the events are created. These results obtain both in the theoretical model and upon calibration to the last half-century of US economic experience. When we consider more general nonparametric preferences and returns, we find that only risk neutral agents pay no attention to extreme events. The results provide a strong motivation for policymakers to ensure full disclosure and dissemination of information regarding probabilities and consequences of extreme outcomes.

Keywords: Attention; Crisis; Endogenous Probability; Extreme Event; Information Cost; Regime Shift

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1 Introduction and motivation

"The collection, digestion, and dissemination in usable form of economic information is one of
the staggering problems connected with our modern large-scale social organization...Specialized
agencies for the supply of information help to bridge the wide gap between what the...business
manager knows...and what he would have to know to conduct his business in a perfectly intel-
ligent fashion.”   Knight (1940), p. 261.

Unexpected economic events can have massive, disruptive effects on a nation. The experience of multiple
crises during the 1990s and 2000s has stimulated researchers’ interest in understanding extreme events in the
US economy. When such events occur, propagation mechanisms may amplify their impact. For example, the
collapse of a major lending institution of course affects its customers, but it may also have macroeconomic
implications for aggregate consumption, investment, and unemployment. Such magnified and correlated
outcomes are interesting not only for their practical relevance, but also economically, since they resemble
the results from a broad class of theoretical studies on herding and strategic complementarities.

The main goal of this paper is to develop a simple economic framework for analyzing rare extreme events,
in order to understand their impact and ramifications. Our model delivers insights into how individuals
respond to extreme events in terms of hedging and asset demands. Interestingly, we find in a parametric
setting that agents may rationally choose to ignore information about extreme events, if this information
is costly. Such a finding ties our work closely to research on information choice and rational inattention,
including Wilson (1975), Sims (2003); Reis (2006); and Veldkamp (2011).

There are two other important areas of research intersection. First there is the recent work on extreme
events, largely in response to the economic crisis. Much of this research analyzes systemic instability.

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1 For evidence on welfare costs of extreme events, see Chatterjee and Corbae (2007), and Barro (2009)
2 See Jaffee and Russell (1997); Barro (2006); Jaffee (2006); Horst and Scheinkman (2006); and Krishnamurthy
3 See Wilson (1975); Bikhchandani et al. (1992); Cooper (1999); and Vives (2008), chapter 6.
4 By extreme, we refer to events that have a high impact on the particular system. This impact can be a financial or
social cost, or of disruption of equilibrium. By rare, we refer to events that are not observed frequently, as in Table 1.
5 See Caballero and Krishnamurthy (2008); Acharya and Richardson (2009); Brunnermeier (2009); Ibragimov,
Jafee and Walden (2009a, 2009b); Reinhart and Rogoff (2009a); Reinhart and Rogoff (2009b); and Shin (2009).
Second, historically there is a long literature examining financial crises and bubbles, in both rational and
behavioral frameworks.\textsuperscript{6} The quantitative models in most of these studies focus on a stationary environment,
although the economic climate is subject to sudden shifts.\textsuperscript{7} In particular, little existing research examines
the economic impact of regime shifts in the probability of encountering extreme events. Our research starts
to rectify this problem by incorporating a simple model of regime shifts in extreme events. We find that the
existence of such shifts may help explain the experience of unhedged extreme events in the US economy,
both theoretically and empirically.

The remainder of the paper is as follows. In Section 2 we review the theoretical and empirical literature on
extreme events. In Section 3 we develop and calibrate a simple model of risky choice, where extreme events
undergo exogenous regime shifts. Section 4 extends this model to endogenous extreme events. Section 5
considers weaker assumptions that allow for unbounded utility, and Section 6 concludes.

\section{Background and related literature}

The paper builds on three strands of related research: extreme events and crises, information choice, and
regime shifts. Regarding extreme events and crises, previous research includes behavioral work such as
Kunreuther and Pauly (2006), who focus on the role of individual myopia in precipitating catastrophes. It
also includes research on bubbles by Blanchard (1979), and Abreu and Brunnermeier (2003), among others.\textsuperscript{8}
There is still no consensus modeling approach for the analysis of extremes. A major challenge is that it is
unclear how individuals behave towards extreme or low probability events. Initial evidence by Allais (1953)
and Kahneman and Tversky (1979) suggested that agents overweight low-probability events. However,
more recent research has uncovered three additional results. First, there is evidence that agents \textit{underweight}
low probability events in realistic situations where they must estimate probabilities based on experience, as
documented by Rabin (2002), Barron and Erev (2003), and Hertwig et al. (2005). Second, statistically, there

\textsuperscript{6}See Fisher (1933); Keynes (1936); Blanchard (1979); Minsky (1982); Friedman and Laibson (1989);
Shleifer and Vishny (1997); Kindleberger (2000); Abreu and Brunnermeier (2003); Allen and Gale (2007); and
Hong et al. (2008).

\textsuperscript{7}Brunnermeier and Sannikov (2011) present an exception with a model that allows episodes of extreme instability.

\textsuperscript{8}Other relevant research includes Jaffee (2006); Ibragimov et al. (2009); and Lorenzoni (2008).
is a bias to underestimate rare events, examined by de Haan and Sinha (1999), and King and Zeng (2001). Third, expected utility may not accurately predict responses to low probability events, a phenomenon studied by Bhide (2000) and Chichilnisky (2000). The finding that agents may systematically underestimate low probability events is particularly interesting, and suggests a systematic lack of knowledge that is not possible to address in current economic frameworks such as robust control and the theory of ambiguity aversion. These frameworks typically presume that agents are aware of their lack of knowledge. By contrast, the most devastating types of rare events involve situations where agents are unaware of their lack of knowledge, which we may term meta-ignorance. A final, promising approach to extremes is the rare disaster literature, pioneered by Barro (2006). The author constructs an economy with the risk of a rare disaster, represented by a large drop in the economy’s wealth endowment. The calibrated model explains the equity premium and low risk free rate puzzles. Gabaix (2008), Gabaix (2010), and Wachter (2011) generalize the Barro framework to account for dynamic probability of extreme events. These latter authors document that their models can explain outstanding macroeconomic and finance puzzles as well as the behavior of stock volatility.

Regarding information choice, work by Morris and Shin (2002), Sims (2003), Reis (2006), Veldkamp and Wolfers (2007), Skreta and Veldkamp (2009), and Veldkamp and Van Nieuwerburgh (2010) shows that agents do not always use all available information. This approach appeals to costs of information processing, so that agents choose to ignore potentially valuable, available information. However, these papers generally do not specify the form and empirical size of costs. The information choice approach has been able to explain a number of anomalies in economics, including the home bias puzzle, asymmetric business cycles, portfolio under-diversification, and ratings inflation. Recent economic experience suggests, moreover, that an important impediment to market performance is lack of knowledge about how to forecast and hedge extreme events. This lack of knowledge reflects non-stationarity of the economic environment, which we embed in our model with the device of regime shifts.

Regarding regime shifts, there is ample evidence that the structure of major economic and financial variables is subject to sharp breaks. Hamilton (1989) develops the modern methodology of regime shifts, and shows

\[^9\text{Negative examples of meta-ignorance include the current financial market crises of fall 2008, climate change, impact of new technology and natural catastrophes. See Bazerman and Watkins (2004) and Taleb (2005). Positive examples could include discovery of North Sea oil in the 1960s.}\]
its applicability to the macroeconomy. In economic and financial markets, evidence of regime shifts is documented by Hamilton and Lin (1996); Ang and Bekaert (2002); Ang and Chen (2002); Ang and Bekaert (2004); and Ang and Bekaert (2005). For theoretical modelling of regime shifts, see Reitz (1988); Evans (1996); Bekaert et al. (2001); Barro (2006); and Angeletos et al. (2007).

2.1 Contributions of our paper

Our paper contributes to the literature in several important ways. First, we examine extreme events by generalizing the well-understood portfolio choice framework based on constant relative risk aversion and lognormal returns. We therefore can exhibit behavior concerning the impact of extreme events in a transparent, rational setting. Second, based on theoretical and empirical considerations, we incorporate latent regime switches in the likelihood of extreme events, which may be exogenous or endogenous. Our paper appears to be the first to analyze the economic impact of extreme events using this framework. Finally, we provide support for the inattention and information choice literature of Sims (2003), Reis (2006), Veldkamp and Van Nieuwerburgh (2010), and Veldkamp (2011), since we give evidence on the size of costs needed to make agents ignore information about important extreme events.

3 Risky choice with exogenous extremes

In this section, we describe the risky choices of an individual faced with rare extreme events. We first present the general setup in section 3.1. Then for the remainder of sections 3 and 4 we specialize to a useful parametric format. We will return to the general framework in section 5. There are three basic ingredients in our parametric setup. First, the base model features a lognormal distribution with constant relative aversion (CRRA) utility. This CRRA-lognormal approach is very tractable and replicates key features of financial
data. Therefore it is commonly used for macroeconomics, portfolio choice and asset pricing, as in the work of Campbell (1994), Campbell (1996), and Campbell and Viceira (2002).\textsuperscript{10}

Second, our framework consists of a single representative agent. This framework allows us to analyze the average behavior when large numbers of similar investors are engaged in risky borrowing.\textsuperscript{11} The representative agent approach is typical of modern finance research in the tradition of Lucas (1978). Third, the case of rare events is handled by a mixture model or regime switch approach. Regime switches have been shown to characterize both economic and financial data, by Hamilton (1989) and Hamilton and Lin (1996). Regime switches are also empirically significant in modelling stock market correlations and variances, as shown by Ang and Chen (2002), Dueker (1997), and Haas et al. (2004). Regime switches have also been utilized to model rare events in financial economics–by Reitz (1988), Evans (1996), Gourieroux and Monfort (2004), and Barro (2006). Finally, the subprime lending innovation starting after 2000 may be considered a regime shift in the risk distribution of the associated mortgage securities.

**Notation and Calibration.** The core notation used in the paper is as follows:

- The quantity $d$ denotes agents’ demand for risky investment, relative to available wealth;
- Superscript $\ast$ denotes an optimum;
- Superscript $T$ denotes a decision or wealth level during typical periods;
- Superscript $E$ denotes a decision or wealth level during extreme periods;
- Subscript $n$ denotes an endogenous investment or wealth level;
- Subscript $x$ denotes an exogenous investment or wealth level;
- $m > 1$ denotes the multiple by which volatility $\sigma$ increases during an extreme event;
- $c > 1$ denotes the multiple by which the likelihood of extremes increases in endogenous extreme events.

\textsuperscript{10}For implementation of the CRRA-lognormal model, see Campbell (1994) p. 469; Campbell (1996) p. 304; and Campbell and Viceira (2002) Chapter 2. Other texts are Huang and Litzenberger (1988); Lyons (2001); and Vives (2008).

\textsuperscript{11}The analysis of many similar investors is also used in literature on strategic complementarities, see Cooper (1999).
3.1 A mixture model setup

We now define the general setup for returns. An investor faces the possibility of extreme events in a risky asset. She has to decide what fraction \( d \) of wealth to invest in the risky asset with returns \( \tilde{r} \) and how much to invest in a riskless asset with predetermined, constant returns \( r_f \). All she knows is that the distribution \( f(\cdot) \) of risky asset returns \( \tilde{r} \) over the next period (a year, say) has either finite or potentially infinite variance.

How can she model this tricky environment? There is a vast statistical literature dating back over 100 years (see Pearson (1894)) on formalizing such situations, where data can potentially come from two or more component distributions \( f_i(\cdot), i \geq 2 \). A standard approach in statistics is to formalize the overall distribution \( f(\cdot) \) with a mixture model, of the form \( f(\cdot) = \sum_{i=1}^{n} \alpha_i f_i(\cdot) \), where \( f_i(\cdot) \) is the \( i \)th component distribution, \( n \) is the total number of components, and \( \alpha_i \geq 0 \) is the weight on the \( i \)th component distribution, \( \sum_i \alpha_i = 1 \).

Practically, mixture models are used to approximate numerous unknown distributions. Indeed, even simple normal mixtures with two or three components can approximate a vast array of distributional shapes with significant skewness and kurtosis (McLachlan and Peel (2000), Chapter 1). Mixture models combine some of the benefits of parametric and nonparametric models, since they can be estimated by maximum likelihood with easily interpreted parameters, and are also flexible because varying the weights can yield many different distribution shapes. For textbook expositions, see Everitt and Hand (1981); McLachlan and Krishnan (1997); and McLachlan and Peel (2000). Economic applications of mixture models are described in the regime-switching literature, see Hamilton (1989); Hamilton (1994); and Kim and Nelson (1999). Similar models are also used in the disaster literature of Barro (2006); Gabaix (2008); and Barro and Ursua (2011). In keeping with this well-established research field, we formalize risky returns with a mixture model in sections 3.4 and 5, below.

3.2 Excessive investment in a risky asset: A benchmark case

Economic research concerning crises often focuses on the aggregate effects of excess borrowing for investment, as discussed by researchers from Fisher (1933) to Allen and Gale (2007). Such excessive borrowing is
sometimes motivated as irrational. While irrationality can certainly drive excess behavior in many settings, it is valuable to determine whether such behavior may arise in a simple, rational framework. We start by showing that such excessive investment may be consistent with rational behavior in a very general setting.

Consider a general neoclassical utility function \( U(W) \) that depends on wealth \( W \).\(^{12}\) Among other qualities, this utility function is strictly increasing, bounded, continuous and concave. Similar to the approach of Campbell and Viceira (2002), the agent is endowed with initial wealth \( W_0 \), and invests a proportion \( d \) in a risky asset with returns \( r = r^f + \varepsilon \). The remainder is invested in a riskfree asset with returns \( r^f \). Thus, \( W = dW_0(1 + r) + (1 - d)W_0(1 + r^f) \), or

\[
W = dW_0(1 + r^f + \varepsilon) + (1 - d)W_0(1 + r^f) \\
= dW_0(1 + r^f) + dW_0\varepsilon + (1 - d)W_0(1 + r^f) \\
= W_0(1 + r^f) + dW_0\varepsilon
\]

We will use the expression for the objective function in (1) for proving the propositions below. The agent maximizes utility subject to the wealth constraint, which is a strictly convex program—that yields a unique solution \( d^* \)– and unique expected wealth \( W^*(d^*) \). We have the following proposition and corollary.

**Proposition 1.** If the investor deviates from the optimal investment strategy \( d^* \) by choosing a suboptimal investment strategy \( \hat{d} \) during a small proportion \( \alpha \) of the time, her expected utility loss is bounded above.

*Proof.* Intuitively, if the investor deviates from the optimum only a discrete proportion of the time and the utility function is bounded, then the expected utility loss is bounded. See the Appendix for a formal proof.

**Corollary 1** If there are high enough costs to learning whether she is behaving suboptimally a small proportion \( \alpha \) of the time, the investor will rationally choose to continue behaving suboptimally.

*Proof.* See Appendix.

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\(^{12}\) By neoclassical utility function, we mean one that is strictly increasing and differentiably continuous, as in Allen and Gale (2007), chapter 2.
Theorem 1 and Corollary 1 show that for standard expected utility functions, if agents are suboptimal some of the time and there are costs to detecting extremes, then agents can rationally choose to be suboptimal.\textsuperscript{13} While this insight is valuable, it is important to be able to quantify the results with observable economic parameters. We provide such calibrations below based on standard parametric utility functions and return processes, to which we now turn.

### 3.3 Base model

We first consider a base model of 'typical' events, where asset returns obey a simple stochastic law. The decision environment consists of a single individual with initial wealth $W_0$, choosing a fraction of wealth $d$ to invest in a risky asset. For these typical economic environments, the investor’s problem is straightforward: she maximizes expected utility by choosing the fraction $d$ to invest in the risky assets. In order to develop the intuition of the previous subsection more concretely, we utilize an important class of preferences and return processes. In particular, we suppose that the investor’s preferences exhibit constant relative risk aversion over wealth $W$, $U(W) = \frac{W^{1-\gamma}}{1-\gamma}$, where $\gamma$ is the coefficient of relative risk aversion. We also assume that the random terms in risky asset returns are lognormally distributed,

$$\tilde{r} \equiv \log(1 + \tilde{R}) \sim N(\mu, \sigma^2).$$  \hspace{1cm} (2)

These classes of preferences and returns are widely used in financial economics, for example Campbell (1996), and Campbell and Viceira (2002). To solve the investor’s problem, observe that the expectation of a lognormal variable $z$ satisfies $\log E(z) = E(\log z) + \frac{1}{2}V(\log z)$. Then, ignoring the constant $1 - \gamma$, and exchanging logs and expectations, we can write the investor’s maximization problem as

$$\max_d \log EW^{1-\gamma} = (1 - \gamma)E(w) + \frac{1}{2}(1 - \gamma)^2V(w),$$

\textsuperscript{13}Similar results appear in the literature on satisficing behavior and procrastination, see Akerlof and Yellen (1985); Akerlof and Yellen (1991); Rabin (2002); and O’Donoghue and Rabin (2008).
subject to \( w = r + w_0 \), where \( w = \log W \), \( r = \log(1 + R) \), and \( w_0 = \log W_0 \). To evaluate the above objective function, we therefore must compute the mean and variance of portfolio returns. The mean excess return is \( E[r - f^f] = d[E(r) - r^f] + \frac{1}{2} d(1 - d)V[r] \). The variance of the portfolio return is \( d^2 V[r] \). Using equation (2), and standard algebraic manipulation as in Chapter 2 of Campbell and Viceira (2002), we can rewrite the investor’s problem as

\[
\max_d d[E(r) - r^f] + \frac{1}{2} d(1 - d)V[r] + \frac{1}{2} (1 - \gamma)d^2 V[r] \quad (3)
\]

where \( \hat{\mu} = [E(r) - r^f] \). Taking derivatives yields first order conditions \( \hat{\mu} + \frac{1}{2} (1 - 2d)\sigma^2 + (1 - \gamma)d\sigma^2 = 0 \), or \( d[\sigma^2 - (1 - \gamma)\sigma^2] = \hat{\mu} + \frac{1}{2} \sigma^2 \). The optimal solution is therefore

\[
d^* = \frac{\hat{\mu} + \sigma^2}{\gamma \sigma^2} = \frac{2\hat{\mu} + \sigma^2}{2\gamma \sigma^2}. \quad (4)
\]

Equations (3) and (4) represent the basic form of objective function and optimum, which we shall use throughout the remainder of this paper. Intuitively, the optimal risky investment is increasing in expected returns and decreasing in risk aversion and variance. Our model assumes that the investor can borrow at the riskfree rate, and abstracts from the possible deadweight costs of default that might arise when the agent chooses to invest with leverage. In section 4.3 below we discuss the possibility of default costs, which would qualify our results.

3.4 A model of exogenous extremes

Now we consider the case of rare extreme events. Following the mixture model discussion of Section 3.1, and literature on peso problems, we model this situation as a small-probability regime switch in risky asset returns. Specifically, the structure of the problem is unchanged from above, except that the risky return now
obeys (2) most of the time, but a small fraction \( \alpha \) of the time there is a regime shift to a period of larger tail events:

\[
\tilde{r} \sim N(\mu, \sigma^2), \text{ with probability } 1 - \alpha \text{ (Typical regime)}
\]

\[
\sim N(\mu, (m\sigma)^2), \text{ with probability } \alpha \text{ (Extreme regime)},
\]

where \( \alpha \) is small, and \( m > 1 \) denotes a multiplicative increase in volatility during extreme periods. This specification allows us to model the notion that asset volatility increases during extreme events, which occur with a relatively small probability in financial markets. While expression (5) permits us to focus on low probability events in a transparent manner, we generalize this approach in Section 5.

We next examine three levels of investor awareness about the stochastic environment: full knowledge of the environment and the specific regime, full knowledge of the environment but stochastic determination of the specific regime, and complete ignorance that an extreme regime may occur at all.

**Agent Knows the Environment and the Specific Regime**

First, consider a situation where the individual knows the stochastic environment as well as which regime holds at each moment. At the beginning of each period, based on the regime that prevails, she solves for the optimal demand in each regime.\(^\text{14}\) Using the same optimization approach as before, the optimal demand will now depend on the regime, and is a vector \( d^* = (d^T, d^E) \), with \( T \) referring to the typical regime and \( E \) to the extreme regime. The typical regime occurs with probability \( 1 - \alpha \) and extreme periods occur with probability \( \alpha \). Therefore the optimal demand vector is

\[
d^T = \frac{\hat{\mu} + \frac{\sigma^2}{2}}{\gamma \sigma^2} = \frac{2\hat{\mu} + \sigma^2}{2\gamma \sigma^2}, \text{ with probability } 1 - \alpha
\]

\[
d^E = \frac{\hat{\mu} + \frac{(m\sigma)^2}{2}}{\gamma (m\sigma)^2} = \frac{2\hat{\mu} + m^2 \sigma^2}{2\gamma m^2 \sigma^2}, \text{ with probability } \alpha.
\]

\(^\text{14}\) The individual does not know the value of risky returns, just the distribution from which they come. Observe that the mixture of log-normals is not restrictive on the unconditional distribution. Conditionally, each regime satisfies log-normality, but unconditionally, a mixture of normals can approximate most empirically observed return distributions arbitrarily closely. For more details on normal mixtures, see McLachlan and Peel (2000).
This is the basic form of investment demand, under parametric utility and exogenous extremes. We will use (6) for this and the next section.

Properties of the Solution The solution in (6) has two distinct properties. First, for positive excess returns \( \hat{\mu} \), it is clear by inspection that investment in the typical regime exceeds the investment in the extreme regime, i.e. \( d^T > d^E \); it is intuitive that more would be invested in the safer typical regime \( d^T \) than in the riskier extreme regime \( d^E \).\(^{15}\) Second, extreme regime investment \( d^E \) depends indeterminately on the volatility multiple \( m \). This is intuitive because as \( m \) rises the extreme regime features higher volatility, which entails the possibility for higher returns, but also higher potential loss. Without further assumptions it is not straightforward to determine the effect of this tradeoff.

A quantitative sense of this differential is achieved by calibrating expression (6) to US data. We use a standard set of calibration values, as displayed in Table 2. Table 3 shows the results of applying these values to expression (6). We find \( d^T \) always greatly exceeds \( d^E \), as expected. For example, with risk aversion \( \gamma = 2 \), we find that \( d^T = 1.73 \) and the maximal \( d^E \) is 0.91. The ratio of \( d^T \) to maximal \( d^E \) is at least 1.90. Thus the demand in a known extreme regime involves no leverage (i.e. \( d^E < 1 \)), and is around 2 times smaller than in the typical regime. This result is qualitatively intuitive, if we think of the extreme regimes as high volatility, disaster periods, where most investors hold small amounts of risky assets, and typical regimes as good or boom periods, when it is relatively more attractive to hold a large position in risky assets.

We also examine another perspective on investors’ risk positions, based on the common propensity of individuals to expand their consumption through borrowing.\(^ {16} \) To measure the actual situation for the US economy, we calculate an empirical version of the investment ratio \( d \) by computing the ratio of total assets

\[\frac{m^2[2\hat{\mu} + \sigma^2] - 2\hat{\mu} - m^2 \sigma^2}{2\gamma m^2 \sigma^2} > 0.\]

The denominator is positive, so this simplifies to \( 2\hat{\mu} m^2 + m^2 \sigma^2 - 2\hat{\mu} - m^2 \sigma^2 > 0 \), or

\[2\hat{\mu}[m^2 - 1] > 0.\]

Since \( m > 1 \) by assumption, the above relation holds if and only if \( \hat{\mu} > 0 \), as was to be shown.

\(^{16}\) This propensity is related to the concept of “over-borrowing,” used by Fisher (1933) in the context of financial crises. For related research see Abreu and Brunnermeier (2003); Lorenzoni (2008); and Shin (2009).
to net worth based on the data for households and nonprofit corporations in the US. The results are illustrated in Figure 1. Evidently, this ratio has tended to increase slightly relative to the first observations in the 1940s and 1950s, and has exceeded unity in every year. Thus, the historical experience of the US economy indicates that $d$ is consistent with a leveraged average investor throughout the last half century.\footnote{We can derive the parameter ranges over which $d^T$ involves leverage. We need to show that $d^T > 1$, or using (6), this means $\mu + \frac{\sigma^2}{\gamma \sigma^2} > 1$. By positivity of $\gamma$ and $\sigma^2$, we can write this as $2\mu + \sigma^2 > 2\gamma \sigma^2$, or $\frac{\mu}{\sigma^2} > \frac{2\gamma - 1}{2}$. Given a risk aversion of 2, for example, this expression says that leverage is optimal when the Sharpe ratio exceeds 1.5.}

**Agent Knows the Environment, but does Not Know the Regime**

Now we consider a situation where the investor knows that there are regime shifts, but does not know, ex ante, which regime will obtain. The investor, as an expected utility maximizer, will choose\footnote{For background on expected utility, see Von Neumann and Morgenstern (1944); Gilboa and Schmeidler (2001) and Gilboa (2009). A non-expected utility maximizer could have a different weighting scheme, see Gilboa (2004).} an intermediate level of demand $\hat{d}$, as shown in the following proposition:

**Proposition 2:** Consider an investor that is an expected utility maximizer with a neoclassical strict monotone increasing utility function $u(d)$, a convex budget constraint, and an environment of regime shifts in extreme events. Denote the optimal risky asset demands during known typical and extreme regimes as $d^T$ and $d^E$, respectively, where $d^E < d^T$. Denote the optimal risky demand when the investor does not know which regime obtains as $\hat{d}$. In this setting, $\hat{d}$ is bounded by $d^T$ and $d^E$, that is, $d^E \leq \hat{d} \leq d^T$.

*Proof.* See Appendix. □

This result is intuitive: if the investor is unsure whether an extreme regime prevails she will reduce her level of investment relative to the typical demand $d^T$, but not as low as if she were certain to be in an extreme regime, $d^E$. The implication is that the investor over-invests when the extreme regime holds and under-invests when the typical regime holds compared to the optimal investment with full information.
Agent Unaware of the Extreme Regime

In the preceding examples, the investor was aware that the environment presented the possibility of extreme risks. By contrast, some of the most significant extreme events in history have been unknown and unforeseen by the public at large.\(^19\) One way to model such ex ante ignorance about extreme regimes is to use a hidden regime shift.\(^20\) Specifically, although the true risky return distribution features a regime shift as in (5), the investor believes that the typical regime always holds, i.e. \(\bar{\varepsilon} \sim N(\mu, \sigma^2)\) with probability 1. Accordingly, she demands \(d = d^T\) with probability 1, instead of probability \(1 - \alpha\) as in equation (6). The investor is therefore over-invested \(\alpha\%\) of the time, investing \(d^T\) instead of the much smaller and optimal \(d^E\).

We may ask two important questions about the investor’s behavior. First, how much does this suboptimal investment hurt her? This question is natural in light of Proposition 1 because the suboptimality only occurs a small percentage of the time. Second, if the investor can learn about the extreme regime at a cost, what is the impact on her investment strategy? We summarize the answers to these questions in Proposition 3 and Corollary 2, below.

**Proposition 3.** The cost to investors of suboptimal behavior during extremes is bounded above by a constant \(K\), which is proportional to squared, standardized excess returns \(\left(\frac{\hat{\mu}}{\sigma}\right)^2\).

**Proof.** See Appendix.

**Corollary 2.** If the costs of learning about extreme events are above a finite threshold, the investor will prefer to over-invest during extreme periods.

**Proof.** See Appendix.

\(^{19}\)In addition to 2008’s financial crisis, other negative examples include the Black Death of 1348; the 1929 US stock market crash; the set of events leading up to the creation of the atomic bomb; global warming; and the devastation of 2005’s Hurricane Katrina. Positive examples include the invention of the wheel; signing of the first US copyright law in 1790; the Wright brothers’ 1903 flight; and the record-breaking US stock market levels of the 1990s.

\(^{20}\)To the best of our knowledge, this formulation of hidden extreme events is novel to the current paper. A parallel framework is used by Gourieroux and Jasiak (2001), who provide an asset demand application, although they do not consider hidden regimes, nor endogenous extremes.
We can see from the Appendix, equation (19), that the constant $K$ is decreasing in risk aversion $\gamma$ and volatility $\sigma$, and increasing in expected excess returns $\hat{\mu}$. This is intuitive because investors who are more risk averse and face more volatility and lower returns will risk a smaller part of their wealth. Consequently, such investors will have less at stake during extreme periods and will therefore face lower costs of suboptimal behavior.

In sum, according to Proposition 3 and Corollary 2, if there are large enough costs to learning about extremes, the investor’s strategy is insensitive to rare extreme events. This is true even when extremes deliver a large effect on return volatility.\textsuperscript{21} To summarize this subsection, we have shown that in an environment of exogenous extremes, a knowledgeable investor may invest much more in normal times than in known extreme regimes. We have also provided a bound on the utility loss from suboptimal behavior by investors who do not understand the economic environment. The existence of this bound is consistent with the literature on global games, rational inattention and information choice.\textsuperscript{22} It suggests that even if agents were informed of the suboptimality of their investment strategy, a high enough level of costs associated with learning about extremes will prevent them from shifting their strategy.

### 3.5 Calibration to the US economy

We calibrate Proposition 3 to US data using equation (19) from the Appendix. The results are displayed in Figure 2. This figure shows that even if volatility doubled ($m = 2$), the costs of maintaining excessive risky investments is bounded from above by a range of around 3% to 24% of wealth, based on the calibrated probability of the extreme regime of 1.7% annually. These costs decrease with risk aversion, because more risk averse investors would always maintain smaller investment amounts. Since U.S. wealth is approximately four times its annual GDP, this range also represents approximately 12% to 96% of GDP. The high values for this range indicate that extreme events represent a material cost for the U.S. economy, and thus

\textsuperscript{21} For related contexts involving biased perception of virgin risks and fearsome risks, see Chichilnisky and Heal (2003); Pavlov and Wachter (2006); Weber (2006); and Sunstein and Zeckhauser (2008).

\textsuperscript{22} See Morris and Shin (2002); Sims (2003); Skreta and Veldkamp (2009); and Veldkamp (2011).
suggest there could be significant social benefits from creating greater information to allow greater and more efficient hedging of such risks.

4 Risky Choice with Endogenous Extremes

The likelihood of extreme and rare events is often affected by the investment behavior of agents in the economy. Endogenous extreme events would include, for example, the effect of human activity on extreme climate changes, and the effect of risky borrowing on financial crises. Accordingly, in this section, we consider a situation where excessive risky borrowing raises the likelihood that the extreme regime will occur. This environment entails more complex information processing for investors, since their returns depend on the likelihood of extremes, which in turn depend on their investment strategies. Similar to the literature on information choice of Sims (2003), such processing costs may lead investors to ignore potentially important information. A further layer of complexity arises when there is complete lack of knowledge, such that individuals are unaware of their collective impact on the likelihood of unforeseen extremes. In light of these considerations, we formalize endogenous extremes by considering an investor who believes the risky return comes from a single distribution as in equation (2), while in truth, the distribution switches endogenously. Optimally the investor should use a cutoff level for risky investment, as we showed in (6). However, unaware of the consequences, she follows the approach of (4) and just chooses the more aggressive investment demand appropriate for the typical regime, namely $d^T$.

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24 The role of excess borrowing in precipitating extreme financial market behavior has been motivated by heightened investor and bank fragility due to lack of liquidity. Prominent examples are the cases of LTCM in 1998 and Lehman Brothers in 2008. Such firms and investors are especially susceptible to even small liquidity shocks and margin calls, see Shleifer and Vishny (1997). Another approach is taken by the research on bubbles and financial crises, see Blanchard (1979) and Allen and Gale (2000).

25 Examples include climate change, or stock market bubbles. This class of extreme events is related to rare events of Taleb (2005), and oblivious ignorance of Bhid (2000). In geopolitics, an instance of unknown endogenous extremes could be the set of events in the early cold war that culminated in the Cuban missile crisis of 1962. This resembles a reverse peso problem: by failing to account for their own ignorance, rational individuals do not anticipate extreme events, which they themselves precipitated.
Once more we may ask two questions. First, how much does this situation harm the investor? In order to answer this question, we compute the expected wealth from behaving optimally and suboptimally. Optimal investment involves a cutoff rule, with potentially non-constant \( d \), while suboptimal investment involves a constant investment rule for \( d \). We are able to compute a bound for the losses that will arise from the suboptimal strategy. Second, under what conditions will there be strong economic incentives for her to learn? If the costs of learning are high enough, a risk averse individual may well ignore endogenous, high-impact regime shifts.

### 4.1 A Two-period model

Substantial evidence indicates that excessive credit and risky borrowing help to create extreme financial events.\(^{26}\) We recognize this evidence by assuming that there are two periods in the economy, with the first-period investment choice affecting the return distribution in the second period. In particular, if the investor is too leveraged in period 1, then the likelihood of extremes is increased to \( c \alpha \) in period 2, where \( c > 1 \) is the multiple by which extremes become more prevalent. Thus, in this endogenous extreme events model, if the investor chooses a leveraged position in period 1, then in period 2 the typical regime occurs with probability \( 1 - c \alpha \) and the extreme regime occurs with probability \( c \alpha \). Using the same approach as in Section 2, the optimal demand vector is similar to that of (6):

\[
\begin{align*}
\hat{d}^{T} & = \frac{2\hat{\mu} + \sigma^2}{2\gamma \sigma^2}, \quad \text{with probability } 1 - c \alpha \\
\hat{d}^{E} & = \frac{2\hat{\mu} + m^2 \sigma^2}{2\gamma m^2 \sigma^2}, \quad \text{with probability } c \alpha.
\end{align*}
\]

We will use this expression to calculate the effect of endogenous extremes on risky behavior.

**Period 1:** We assume that period 1 is a typical regime. If the investor were unaware of the impact of her period 1 investment on period 2 volatility, she would then simply choose the optimal demand for a typical period, namely \( \hat{d}^{T} \); we now assume that this decision exhibits leverage and refer to it as the leveraged (\( L \))

\(^{26}\)See Fisher (1933); Bernanke (1983); (Allen and Gale (2007); Lorenzoni (2008); and Shin (2009).
outcome, that is, \( d^L > 1 \). On the other hand, if the investor is aware that her investment decision in the first period may have a negative impact on the second period distribution, then she may wish to temper the size of her first period investment. In particular, we will assume that as long as the period 1 investment does not exceed 1.0, that is, it does not involve leverage, then the extreme regime will occur with probability \( \alpha \), and not \( c\alpha \). We will describe an investment decision as prudent when \( d = 1 \), and denote it as \( d^P = 1 \).

In this setting, the first period investment choice must consider the impact it will have on the second period volatility. She can invest \( d^P = 1 \), which has the benefit of ensuring a lower probability of the extreme regime in the following period, but at the cost of foregone returns; or she can borrow to invest \( d^L > 1 \), which has the benefit of higher expected returns in period 1, but at the cost of an increased danger of the extreme regime.

**Period 2:** In the second period, the probability of extremes is

\[
\Pr(\text{extremes}) = \begin{cases} 
\alpha, & \text{if investor choose } d^P \text{ in the first period,} \\
\alpha c, & \text{if investor choose } d^L \text{ in the first period.}
\end{cases}
\]

Since period 2 is the last period in the planning horizon, we assume the investor cannot take on a further leveraged position with \( d^L > 1 \). Instead, the investor may either invest \( d^P = 1 \) for period 2–what we are calling the prudent level–or invest an even smaller amount \( d^E < 1 \) that would be optimal if the extreme regime occurs in period 2.

In analyzing behavior in the 2-period model, as earlier, we consider two levels of investor awareness of the economic environment, corresponding to complete understanding and complete misunderstanding.

**Agent Knows the Environment and the Specific Regime**

In this case, the representative investor understands that the environment features regime shifts in the likelihood of extreme events. Further, she knows that leverage raises the likelihood of the extreme regime. We summarize the result in Proposition 4.
**Proposition 4.** *The expected net benefit of leverage (d > 1) in period 1 is bounded, for investors who know that the environment features regime-switching in extreme events. This effect depends on the values of \( \hat{\mu} \) and \( \sigma^2 \).*

*Proof.* See Appendix.

**Agent Unaware of the Extreme Regimes**

In this case, the investor does not know that there are regime shifts and does not know that she can influence the likelihood of extremes. Therefore, in period 1 she always selects the optimal investment for a typical regime, namely \( d^L > 1 \), and in period 2 she invests as much as she can, namely \( d^P = 1 \). Nevertheless, even in this circumstance, the effect resulting from the suboptimal investment decision is bounded. We summarize this result in Proposition 5.

**Proposition 5.** *The expected net loss from suboptimal investment is bounded, for an investor who does not understand that the environment features regime-switching in extreme events.*

*Proof.* See Appendix.

Proposition 4 above has shown that rational investors may knowingly increase the likelihood of extreme events in the second period. Proposition 5 now shows that investors who do not understand the environment face possible losses that are bounded. Therefore if the costs of learning about the environment are large enough, investors may choose to continue with a socially suboptimal strategy.

### 4.2 Calibration to the US economy

We calibrate Propositions 4 and 5 to the US data shown in Table 2—using the expressions for the net benefits and costs from the Appendix.\(^{27}\) The results are displayed in Figures 3 and 4. Figure 3 shows the effect of having leverage in the first period for an investor who understands that this will increase the likelihood of

\(^{27}\)The expressions for Propositions 4 and 5 are in Equations (30) and (33).
extreme events. In this case, there is a moderate net benefit of leverage, between $-5\%$ and $18\%$ of wealth. Figure 4 shows the net effect for an investor who does not understand the extreme regimes and is levered ($d^L > 1$) in period 1, and fully invested in period 2. We see that there are moderate costs from not reducing investment below wealth. Even with a large risk aversion of 5 and a 5-fold increase in both volatility $m$ and extreme likelihood $c$, these costs represent between 0.19\% to 10\% of available wealth. Thus, whether the investor was aware or unaware of endogenous regime shifts, for the calibrated US economy over the last 50 years it may not have been viable to suggest that she limit investment below available wealth. The reason is that the extreme event represents an externality that raises the cost of investing in the future, with little or no impact on the benefits of taking advantage of high returns in the present. Therefore it is necessary to provide incentives for investors to internalize the costs of future extreme events.

4.3 Some salient issues

The preceding setup represents a specific approach to treating the effect of extreme events. While the results are instructive, there are a number of issues that may be raised. We briefly outline these issues and our attempts to deal with them below, before exploring a more general approach in Section 5.

*Mixture Model Assumption.* Thus far we have utilized a mixture of normal distributions to model returns. This may seem to be an unusual choice, since the normal distribution is not heavy tailed. Our response to this issue is twofold. First, as noted since Pearson (1894), a mixture of normals can capture highly skewed shapes, multimodal behavior, kurtosis, and other non-normal features. Second, in the next section we demonstrate the possibility of fat-tailed data in a transparent manner using a more general mixture model. In particular, we generalize equation (5) to be a mixture of a finite variance and infinite variance distribution, in equation (10) below.

*Existence of Expected Utility.* Another issue with our framework thus far is that in some situations expected utility might not exist, as shown in Geweke (2001). In order to handle unbounded expected utility, in Section

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5 we use more general preferences, characterized only in terms of the sign of risk aversion—whether agents like or dislike large variance in returns. Our results from this framework are presented in Proposition 6 of Section 5.

Cost of Ignoring the Extreme State. Another way in which unbounded expected utility shows up concerns our results in Proposition 1. That result did not account for very fat tailed events, when expected utility can be unbounded for an investor with leverage. In such a case the cost of ignoring the extreme state will become unbounded. This is similar to the previous issue and we account for it in a similar way, using general utility functions in Section 5. We therefore accommodate infinite costs of ignoring the extreme state. We derive results in terms of risk appetite, which are stated in Corollary 3 of Section 5.

Bankruptcy. One issue on which we have said little concerns investor bankruptcy in the face of leverage. We outline one approach that allows us to integrate bankruptcy considerations into our framework. Suppose that investors have access to leverage. During normal times, a fraction \( \delta_0 \) of investment loans are defaulted on, while during extreme events a fraction \( \delta > \delta_0 \) is defaulted on. Lenders know the default rate is higher during extremes, and therefore charge a premium such that their expected costs are covered. That is, the investor cannot borrow at the risk-free rate, she must pay a percentage fee \( p > r_f \) when borrowing for investment purposes. This fee is assumed to be actuarially fair, i.e. \( p = E[\delta] \). Thus, the investor’s random wealth \( \tilde{w} \) now includes a cost that reflects lenders’ default premium \( p \):

\[
\tilde{w} = d(1 + r - p) + (1 - d)(1 + r_f),
\]

In order to account for bankruptcy risk, one can use equation (8) for the investor’s wealth constraint. This expression could be used to replace the constraint (9) in Section 5 with straightforward results.

Threshold for Attentiveness. One potential weakness of Proposition 3 is that we do not know a lot about the threshold where awareness becomes worthwhile. In particular, if the constant of proportionality \( \theta \) is large, this makes the bite of the proposition less sharp. We approach this issue in two ways. First, from the proof in the Appendix, equation (19), the constant depends on risk aversion \( \gamma \) and the volatility multiple \( m \), which measures the increase in return variance during extremes. We calibrate Proposition 3 for various levels of
in Figure 2. Indeed, the results show substantial diversity—across various specifications the bound can be less than 1% or as large as 80%. This large degree of uncertainty illustrates a need for more information on the structure of preferences and behavior of returns during extreme events. Second, we seek to move beyond the parametrized approach, and therefore model returns with a more general mixture model in Section 5, equation (10). The price we pay for the more general approach is a great deal of indeterminacy about the optimal strategies, as shown in Proposition 6 and Corollary 3 below.

Do Agents Pay More or Less Attention to Extremes? Our previous setup suggests that agents pay less attention to extreme events. One could plausibly argue that nowadays extreme events are more heavily reported and receive greater media attention. Therefore it is not clear whether the appropriate benchmark should be attention or inattention to extremes. This is a subtle issue, which we approach in Section 5 by invoking two key aspects of our model, namely investor reaction to extreme events in terms of preferences, and investor awareness or estimation of the probability of extremes. Regarding investor reaction to extreme events, the main distinction is whether investors’ utility functions respond to infinite variance of wealth during extreme states, i.e. whether investors are risk-averse/loving, or risk neutral. Corollary 3 below shows that risk-averse and risk-loving investors pay attention to extremes, while risk-neutral investors will not pay attention to extremes. Therefore, whether agents pay more or less attention to extreme events becomes an empirical question of whether more agents are risk-averse/loving or risk neutral.

Regarding investor awareness, investors may differ in their knowledge of the possibility for extreme events. Investors may be unaware that the extreme regime exists, or they may be aware that it exists and have to estimate the likelihood of extremes $\alpha$. The case of unawareness is summarized below in Corollary 5: in an environment of extremes with unawareness, risk-averse investors have a massive negative shock to utility, risk-loving investors experience a large utility windfall, and risk-neutral investors experience no effect. The case of awareness is in Proposition 7 and Corollary 4: as long as investors estimate some positive $\hat{\alpha} > 0$ for the likelihood of extremes, even if it is biased, their investment strategies are exactly as if they knew the true extreme likelihood exactly. This result is interesting because it is well known in statistics and
In Section 5 we show (Corollary 4) that such estimation biases are irrelevant as long as investors are open-minded enough to allow for the possibility that the extreme regime exists. From this perspective, therefore, it is moot whether agents’ expectations of extreme events are “rational” or biased.

What if Other Agents Exploit Rational Inattention? We showed in previous sections that agents may be rationally inattentive, using a representative agent framework. Suppose, by contrast, that agents are heterogeneous in their levels of attention. A natural question arises that if some agents are not inattentive, they could profit from the inattentive ones by taking opposite trading positions. While the issue of heterogeneity in attention is beyond the scope of the paper, our setup in Section 5 offers a few suggestions. In particular, all risk-averse and risk-loving agents are very sensitive to extreme events. They will therefore not be exploited because they are very attentive to any information about the stock market. The only agents who will be inattentive are risk-neutral investors. Since these latter investors’ preferences are not affected by wealth variance, they are not exploitable in terms of expected utility. We now turn to Section 5, which formally addresses these issues in a general framework.

5 A model with weaker assumptions

Our preceding analysis focused on a particular parametric setup. We now extend this framework to consider weaker assumptions on data and preferences, in order to deliver more general, credible results. We draw on two literatures, mixture models and partial identification. In order to ameliorate the issue of strict assumptions on utility functions and data, we build on the partial identification literature, summarized in Manski (2007). Specifically, we only consider utility functions to have monotone properties, increasing or

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29 Statistical estimates of the likelihood of extremes are biased the closer to the center of the data are the sample, see de Haan and Ferreira (2006). Experimental evidence shows that when individuals have to learn about the likelihoods, they tend to focus on recent events and underestimate the likelihood of extremes, see Hertwig et al. (2005). The peso problem literature suggests that individuals overestimate the likelihood of extreme events, which may not occur in-sample, see Evans (1996); Reitz (1988); Barro (2006). The literature on survival bias suggests that estimates of the likelihood of important financial market events are biased because many stock markets no longer exist, see Brown et al. (1995).
decreasing in the variance or wealth. In order to model data with potentially infinite variance, we use general mixture models, as in McLachlan and Peel (2000). In the interests of space, we sketch the basic framework in terms of Propositions and Corollaries below.

5.1 Framework for investor facing extreme events

Recall the framework from Section 3.1 where the investor faces the possibility of extreme events in a risky asset with returns $\tilde{r}$. She wishes to invest a fraction $d$ of wealth in the risky asset, the remainder in a riskless asset with returns $r_f$. All she knows is that the distribution $f(\cdot)$ of risky asset returns $\tilde{r}$ over the next period has either finite or potentially infinite variance. We now generalize expression (5) and revisit the investor’s problem.

As discussed in Section 3.1, starting with Pearson (1894), this problem can be shown to be amenable to a mixture interpretation. The overall distribution $f(\cdot)$ is written in terms of component distributions $f_i$. That is, $f = \sum_{i=1}^{\infty} \alpha_i f_i(\cdot)$, which we develop below. At the end of the investment period the investor’s wealth is $\tilde{w}(\tilde{r})$, which depends on returns:

$$\tilde{w} = d(1 + \tilde{r}) + (1 - d)(1 + r_f),$$

(9)

and her preferences are described by a continuous utility function $U(\tilde{w}) : R \rightarrow R$. Let the likelihood of the finite variance regime be $1 - \alpha$ and the likelihood of infinite variance$^{30}$ be $\alpha \in [0, 1]$. Then, as in the mixture model literature discussed in Section 3.1 and above, risky asset returns may be written in the form

$$\tilde{r} \sim f_1(\mu_1, \sigma^2), \quad \text{with probability } 1 - \alpha \quad \text{(Normal Regime)}$$

$$\sim f_2(\mu_2, \infty), \quad \text{with probability } \alpha \quad \text{(Extreme Regime)},$$

(10)

where $\sigma^2 < \infty$ is the variance of risky returns during normal times, $\mu_1 < \infty$ and $\mu_2$ denote the mean returns during normal and extreme times, and $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ denote the probability density function of

$^{30}$The infinite variance distribution could be Student-t, for example.
returns during normal and extreme times, respectively. Now the investor’s problem is to choose the fraction \( d \in [0, 1] \) of her initial wealth to invest in the risky asset. Formally, her problem is

\[
\max_d EU(\tilde{w}) \text{ subject to } \tilde{w} = d(1 + \tilde{r}) + (1 - d)(1 + r_f) .
\]

(11)

**Solution to the Investor’s Problem under Complete Awareness.** If the investor knew which regime prevailed, she could in principle maximize expected utility during each period separately. For example, if her utility function \( U \) is concave increasing in mean wealth and decreasing in variance of wealth, her optimal allocation \( d^* \equiv \arg \min EU(\tilde{w}) \) is typically a well-defined\(^{31}\) vector \( d^* = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \): in normal times she invests some \( d_1 \in [0, 1] \) in the risky asset, and in extreme times, she invests as little as she can, \( d_2 = 0 \). However, since the investor does not know the regime, the problem is more complicated. In particular, the investor’s expected utility is *unbounded* in many plausible settings, for example when utility is a monotone function of the wealth variance (See Geweke (2001) for an example with power utility). In these general settings it turns out to be important to consider risk tolerance and investor awareness, in order to determine optimal investment in the risk asset. We discuss risk appetite and awareness in the two following sections.

### 5.2 Impact of risk appetite

There are three interesting cases of investor preferences: risk-loving, risk averse, and risk-neutral. A risk-loving person’s utility function increases with variance of wealth. Such preferences yield unbounded positive utility in the extreme regime, hence the optimal \( d_2 \) is as large as possible, \( d_2 = 1 \) (or \( d_2 = \infty \) if unlimited leverage is allowed). Conversely, a risk-averse individual’s utility function decreases with wealth variance, yielding infinitely negative utility during extreme periods. For these latter preferences, \( d_2 = 0 \). Finally, a risk-neutral investor is unaffected by infinite wealth variance during extreme periods, and would choose any \( d_2 \in [0, 1] \). Therefore, an investor faced with possible extreme events has a wide range of optimal

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\(^{31}\) Existence of the optimum is guaranteed in the normal regime because the problem involves maximizing a continuous function on a convex set. This follows from the Weierstrass theorem, Simon and Blume (1994), chapter 30.
risky investment strategies. If short-sales and leverage are allowed to levels $d_s < 0$ and $d_l > 1$, then the investor’s optimal investment range will be $d_2 \in [d_s, d_l]$, which is wider than $[0, 1]$. Without knowing more about the investor’s risk preferences, one cannot point-identify what her actual strategy will be. These results are presented in Table 5 and Figure 5. In order to formalize the implications for investor attention, we summarize these results in the following proposition.

**Proposition 6.** Suppose a risk-averse investor faces extreme events as described in (10), and currently invests $d^* = 0$ in the risky asset, receiving finite returns and wealth $W^*$. If she deviates from $d^*$ by choosing an alternative investment strategy $\hat{d} > d^*$ even a small proportion $\theta$ of the time, her expected utility loss is potentially unbounded. Conversely, a risk-loving investor’s expected gain is potentially unbounded.

**Proof.** See Appendix.

**Corollary 3.** Suppose there is a small likelihood $\theta > 0$ of extreme events, and that investor preferences respond to risk, i.e. the investor is either risk-averse or risk-loving. If the investor can choose between investing immediately or first learning about the likelihood of extremes, she will prefer to incur any finite costs to learning about extreme events, before investing in the risky asset.

**Proof.** See Appendix.

Proposition 6 and Corollary 3 show that in a general environment of extremes, as long as preferences are monotone increasing or decreasing in expected wealth variance, investors are either extremely conservative (staying out of the market completely) or liberal (investing all their wealth in the market). The only possibility for diversification is if investors are risk neutral. Irrespective of costs to detect extremes, there is indeterminacy, which largely depends on investors’ risk preferences. This is an important result, because it says that as long as agents have a systematic reaction to risk, during extreme events there is no desire for diversification. Moreover, in the face of extreme events, information is unboundedly valuable and only risk-neutral investors will be rationally inattentive.
5.3 Impact of awareness

In addition to risk preferences, we have to consider the level of investor awareness, which we summarize by their estimates $\hat{\alpha}$ of the likelihood of extremes. If investors are completely unaware that the extreme regime exists, they set $\hat{\alpha} = 0$. If they are aware, they choose $\hat{\alpha} > 0$. We first examine their behavior under awareness, then under unawareness.

**Proposition 7.** Consider an investor with current expected wealth $W^* < \infty$ who decides how much to invest in the risky asset, and who is aware that there may be an extreme regime as in (10). The investor does not know the true likelihood of extremes $\alpha^*$, and therefore has to estimate it. Even if her estimate is biased, as long as it is positive, her optimal risky demand will be the same as if she knew the true likelihood of extremes with certainty.

**Proof.** See Appendix.

**Corollary 4.** Suppose an investor is open-minded that there is a positive likelihood of extreme events, and chooses a positive, potentially biased, estimate of extreme likelihood, $\hat{\alpha} > 0$. Then only if she is risk-neutral or risk-loving will she will invest in the risky asset.

**Proof.** See Appendix.

We summarize what happens if the investor is unaware that regimes exist, in Corollary 5.

**Corollary 5** Suppose an investor currently invests in the risky asset, i.e. $d > 0$. If she is unaware that the extreme regime exists and has risk-averse (risk-loving) preferences, she will subsequently incur infinite utility losses (gains), while a risk-neutral investor will face zero utility losses.

**Proof.** See Appendix.

In economic terms, Proposition 7 says that as long as agents have some open-mindedness that extremes exist, they will invest as if they knew the true likelihood of extremes. An important implication is that when investors are risk-averse, the type of error that affects market liquidity during extreme events is not slightly biased estimates of the likelihood of extremes, it is incorrectly assuming a zero likelihood of extremes.
The reason is that as long as the estimate of extreme likelihood is positive, the effect on expected utility is infinite. It is infinitely negative for risk averse investors, and infinitely positive for risk-loving investors. This effect is exactly the same as if they had perfect estimates of extreme likelihood. For risk neutral investors, the utility effect is always zero so their choices are again unaffected. Turning to Corollary 4, this predicts that if agents believe extreme events are possible, then the only active market participants will be either risk-loving or risk neutral. Finally, according to Corollary 5, Unawareness of the existence of extreme regimes affects investors differentially, depending on their risk tolerance. Under unawareness, risk-averse investors experience massive utility losses, risk-lovers enjoy tremendous utility windfalls, and risk-neutral investors are unaffected during extreme regimes. To summarize this Section’s results, in a general setting of preferences and returns, risk appetite becomes the dominant consideration.

Interestingly, Proposition 6 and Corollary 3 establish substantial indeterminacy for investment strategies in an environment of extreme events. Specifically, the optimal fraction of wealth $d$ invested in the risky asset can be any number in the range $[0, 1]$, depending on investor preferences toward risk. When leverage and short-sales are allowed, the range grows even larger. Our results in this section use a menu of weak assumptions on preferences and returns—preferences include risk aversion, risk neutrality, and risk loving; and returns are represented by a mixture model that has one component with finite variance and the other with infinite variance. We see that as long as agents have utility functions that respond to variance, they will invest nothing (risk-averse case) or everything (risk-loving case) in the risky assets. This is because even a slight chance of extreme events has infinite impact on expected utility of risk-averse and risk-loving investors, as in the disaster literature of Barro (2006); Gabaix (2008); and Barro and Jin (2011). The only possibility for diversification comes with investors who are risk-neutral. We also show in Corollary 3 that, because of the impact on expected utility of wealth, learning about extreme events is important to both risk-averse and risk-loving investors. The only investors who do not care to pay for information about extremes are risk-neutral investors.
6 Conclusions

In this paper we construct a simple latent regime-switching model of portfolio choice, in order to assess the implications for over-investing. Motivated by theoretical and empirical considerations, we examine the benefits and costs of leverage, and of suboptimal investment. We approach these issues in both parametric and more general nonparametric settings. For the parametric setting, our most striking finding is that in both one and two-period models, the benefits of sustained optimal investment are bounded. Thus, investors may knowingly ignore or exacerbate the likelihood of extreme events, especially if there are costs to learning the structure of the financial environment. We also document from our calibrated results that, in the typical regime, investors benefit from leverage, that is from an investment amount exceeding their wealth. Moreover, in the calibration for a two period model, the net costs of suboptimal investments are quite moderate, less than 18% of investor wealth. Overall, we thus document that the costs of ignoring extreme events are quite moderate and the benefits of leverage are substantial.

When we examine a broad nonparametric setting for preferences and fat-tailed returns, we find substantial indeterminacy for investment strategies. The optimal risky investment fraction can be any number in the range $[0, 1]$, depending on investor preferences toward risk. Moreover, as long as agents care about return variance, they will invest nothing (risk-averse case) or everything (risk-loving case) in the risky assets. This result holds because a slight chance of extreme events has infinite impact on expected utility of risk-averse and risk-loving investors. The only possibility for diversification comes with investors who are risk-neutral. Interestingly, learning about extreme events is important to both risk-averse and risk-loving investors. The only investors who will not pay for information about extreme events are risk-neutral.

Our paper therefore provides both a theoretical framework for examining extreme events, and empirical evidence on the scope of the benefits and costs related to learning about extremes. From a policy perspective, our results provide support for research that will provide more complete information on the likelihood and consequences of extreme and rare events. Indeed, if the losses from extreme outcomes create negative social externalities, there may even be a case for subsidizing the creation of information on the causes and consequences of the extreme events.
References


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Table 1: Examples of Extreme and Rare Events

<table>
<thead>
<tr>
<th>Non-Extreme</th>
<th>Frequent</th>
<th>Rare</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Small Impact)</td>
<td>No war, post-1990 western Europe</td>
<td>↓ CO$_2$ pollution</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Multi-nation war, post-1990 western Europe</td>
</tr>
</tbody>
</table>

| Extreme           | ↑ CO$_2$ Pollution                           | Multi-country stock market crash, post-Great Depression |
| (Large Impact)    |                                              | |

Table 2: Calibration to US data

<table>
<thead>
<tr>
<th>Variable</th>
<th>Calibrated Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualized excess stock return $\hat{\mu}$</td>
<td>$\hat{\mu} = 0.081 - 0.009 = 0.072$</td>
<td>Campbell (2003) p. 805</td>
</tr>
<tr>
<td>Annualized stock market volatility $\sigma$</td>
<td>$\sigma = 0.156$</td>
<td>Campbell (2003) p. 805</td>
</tr>
<tr>
<td>Average borrowing rate $r_b$</td>
<td>$r_b = 1 + R_b$</td>
<td>Federal Reserve Bank of St. Louis</td>
</tr>
<tr>
<td>Discount factor $\beta$</td>
<td>$\beta = 0.99$</td>
<td>Mehra and Prescott (2003) p. 907</td>
</tr>
<tr>
<td>Risk aversion $\gamma$</td>
<td>$\gamma \in {1, 2, \ldots, 10}$</td>
<td>Lewis (1999) p. 576</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mehra and Prescott (1985) p. 154</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Mehra and Prescott (2003) p. 907</td>
</tr>
<tr>
<td>Annualized likelihood of an extreme event $\alpha$</td>
<td>$\alpha = 0.017$</td>
<td>Barro (2006) p. 837</td>
</tr>
</tbody>
</table>

We compute the borrowing rate $r_b$ as the average of the monthly Prime Bank Loan rate from January 1949 to December 2008.
Table 3: Risky Asset Demand in Extreme and Normal Times

The table presents risky demand $d^E$ and $d^T$ during extreme and normal times respectively, using equation (6). Demand is computed as a fraction of investor wealth. The calibration is as in Table 2. The parameter $\gamma$ denotes the coefficient of relative risk aversion, and $m$ denotes the volatility multiple, i.e. the increase in volatility during extreme periods.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$d^T$</th>
<th>$d^E$</th>
<th>$d^E$</th>
<th>$d^E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.4586</td>
<td>1.8149</td>
<td>1.4661</td>
<td>1.2396</td>
</tr>
<tr>
<td>2</td>
<td>1.7293</td>
<td>0.9075</td>
<td>0.7330</td>
<td>0.6198</td>
</tr>
<tr>
<td>3</td>
<td>1.1529</td>
<td>0.6050</td>
<td>0.4887</td>
<td>0.4132</td>
</tr>
<tr>
<td>4</td>
<td>0.8646</td>
<td>0.4537</td>
<td>0.3665</td>
<td>0.3099</td>
</tr>
<tr>
<td>5</td>
<td>0.6917</td>
<td>0.3630</td>
<td>0.2932</td>
<td>0.2479</td>
</tr>
<tr>
<td>6</td>
<td>0.5764</td>
<td>0.3025</td>
<td>0.2443</td>
<td>0.2066</td>
</tr>
<tr>
<td>7</td>
<td>0.4941</td>
<td>0.2593</td>
<td>0.2094</td>
<td>0.1771</td>
</tr>
<tr>
<td>8</td>
<td>0.4323</td>
<td>0.2269</td>
<td>0.1833</td>
<td>0.1550</td>
</tr>
<tr>
<td>9</td>
<td>0.3843</td>
<td>0.2017</td>
<td>0.1629</td>
<td>0.1377</td>
</tr>
<tr>
<td>10</td>
<td>0.3459</td>
<td>0.1815</td>
<td>0.1466</td>
<td>0.1240</td>
</tr>
</tbody>
</table>

Figure 1: Asset to Net Worth Ratio for US Households

The figure shows the ratio of total US household assets to net worth. All variables are available from Flow of Funds accounts at the Board of Governors Federal Reserve Bank. The frequency is annual, and the time period is 1945 to 2009.
Figure 2: Net Cost of Excess Investment for a US Investor who faces Exogenous Extreme Events

The figure calibrates the net expected cost of over-investing during exogenous extreme events, as discussed in Section 3. The figure is computed from Proposition 3, equation (19), using US data and the calibration in Table 2. \( m \) denotes the multiple by which volatility increases in an extreme event. The data values displayed in the figure are given explicitly in Table 4. The bound shows how much an investor would have gained as a percentage of wealth, by investing a smaller, optimal amount during rare extreme events that occur with fixed probability \( \alpha \) in a one-period model. According to Corollary 2 this bound may also be interpreted as the minimum information cost to dissuade investors from learning about regime shifts.
Figure 3: Net Benefit from Leverage for Investor who Understands Endogenous Extreme Events

The figure calibrates the net benefit of leverage when the investor knows that excess leverage may raise the likelihood of extreme events by a multiple $c$ in period 2, as discussed in Section 4. The figure is computed from Proposition 4, equation (30). We use US data and the calibration values from Table 2. Further examples of data values displayed in the figure are given explicitly in Table 4.
Figure 4: Net Cost of Suboptimal Investment for Investor who is Unaware of Endogenous Extreme Events

The figure calibrates the net cost of investing all her wealth ($d^P = 1$) in period 2, when the investor is unaware of the possibility of endogenous extreme events in this period. The figure is computed from Proposition 5, equation (33). We use US data and the calibration values from Table 2. $c$ is the probability multiple, i.e. the increase in likelihood of extremes. $m$ is the volatility multiple, i.e. the increase in volatility during extreme periods. Further examples of data values displayed in the figure are given explicitly in Table 4. The surface may be interpreted as the minimum cost to dissuade investors from learning about regime shifts in the likelihood of rare events, as discussed in Section 4.
Table 4: The Benefits and Costs of Leverage during Extreme Regimes

The table presents the benefits and costs of leverage and suboptimal demand as a percentage of investor wealth. \( c \) is the probability multiple, i.e. the increase in likelihood of extremes. \( m \) is the volatility multiple, i.e. the increase in volatility during extreme periods. The parameter \( \gamma \) denotes the coefficient of relative risk aversion. The calibration corresponds to that in Propositions 3, 4 and 5, as displayed in Figures 2 to 4 above.

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>Cost of Over-Investment, ( m=1.5 )</th>
<th>Cost of Over-Investment, ( m=2 )</th>
<th>Benefit of Leverage, ( m=1.5 )</th>
<th>Benefit of Leverage, ( m=2 )</th>
<th>Net Cost of Suboptimal Investment, ( m=1.5 )</th>
<th>Net Cost of Suboptimal Investment, ( m=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0740</td>
<td>0.2296</td>
<td>-0.0598</td>
<td>-0.0570</td>
<td>0.0006</td>
<td>0.0001</td>
</tr>
<tr>
<td>2</td>
<td>0.0370</td>
<td>0.1198</td>
<td>-0.0375</td>
<td>-0.0373</td>
<td>0.0000</td>
<td>0.0005</td>
</tr>
<tr>
<td>3</td>
<td>0.0247</td>
<td>0.0799</td>
<td>-0.0122</td>
<td>-0.0123</td>
<td>0.0004</td>
<td>0.0017</td>
</tr>
<tr>
<td>4</td>
<td>0.0185</td>
<td>0.0599</td>
<td>0.0139</td>
<td>0.0140</td>
<td>0.0011</td>
<td>0.0031</td>
</tr>
<tr>
<td>5</td>
<td>0.0148</td>
<td>0.0479</td>
<td>0.0402</td>
<td>0.0408</td>
<td>0.0019</td>
<td>0.0046</td>
</tr>
<tr>
<td>6</td>
<td>0.0123</td>
<td>0.0399</td>
<td>0.0667</td>
<td>0.0679</td>
<td>0.0027</td>
<td>0.0062</td>
</tr>
<tr>
<td>7</td>
<td>0.0106</td>
<td>0.0342</td>
<td>0.0933</td>
<td>0.0951</td>
<td>0.0035</td>
<td>0.0078</td>
</tr>
<tr>
<td>8</td>
<td>0.0092</td>
<td>0.0300</td>
<td>0.1199</td>
<td>0.1224</td>
<td>0.0044</td>
<td>0.0094</td>
</tr>
<tr>
<td>9</td>
<td>0.0082</td>
<td>0.0266</td>
<td>0.1465</td>
<td>0.1498</td>
<td>0.0053</td>
<td>0.0110</td>
</tr>
<tr>
<td>10</td>
<td>0.0074</td>
<td>0.0240</td>
<td>0.1732</td>
<td>0.1772</td>
<td>0.0062</td>
<td>0.0126</td>
</tr>
</tbody>
</table>

Table 5: Implied Investor Demand for Different Combinations of Returns and Preferences

The table presents the broad implications of different combinations of assumptions on asset returns and investor preferences as in section 5.2. The environment features normal and extreme returns as described in equation (10).

<table>
<thead>
<tr>
<th>Preferences</th>
<th>Typical Regime</th>
<th>Extreme Regime</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Averse</td>
<td>Singleton ( d \in [0, 1] )</td>
<td>( d = 0 )</td>
</tr>
<tr>
<td>Risk Neutral</td>
<td>Any ( d \in [0, 1] )</td>
<td>Any ( d \in [0, 1] )</td>
</tr>
<tr>
<td>Risk Loving</td>
<td>Singleton ( d \in [0, 1] )</td>
<td>( d = 1 )</td>
</tr>
</tbody>
</table>
Figure 5: Examples of Investor Demand Under General Model of Extreme Returns

The figure shows examples of investor demand in a general model of asset returns and investor preferences, as described in section 5.2. The environment features normal and extreme returns as described in equation (10). We present one level of investment for the risk neutral investor, whose optimum is any level in [0, 1].
A Proofs of Propositions

Proposition 1. If the investor deviates from the optimal investment strategy \(d^*\) by choosing a suboptimal investment strategy \(\hat{d}\) during a small proportion \(\alpha\) of the time, her expected utility loss is bounded above.

Proof. We need to show that the expected utility loss \(\Delta EU\) satisfies \(\Delta EU \leq K\), for some \(K < \infty\). First, let us denote the suboptimal wealth level \(\hat{W}(\hat{d})\). Now note that the expected utility loss is the difference between optimal utility and the suboptimal utility that occurs with probability \(\alpha\). Thus \(\Delta EU \equiv U(W^*) - [\alpha U(\hat{W}) + (1 - \alpha)U(W^*)]\), where we drop the argument in \(W()\) for simplicity. Computing the expected utility loss, we obtain

\[
\Delta EU = U(W^*) - [\alpha U(\hat{W}) + (1 - \alpha)U(W^*)] = \alpha[U(W^*) - U(\hat{W})].
\]

By boundedness of the utility function, the quantity in (12) is finite and bounded above, for example, by \(\alpha U(W^*)\). Thus, for \(K = \alpha U(W^*)\), we have that \(\Delta EU\) satisfies \(\Delta EU \leq K\), as was to be shown.

Corollary 1. If there are high enough costs to learning whether she is behaving suboptimally a small proportion \(\alpha\) of the time, the investor will rationally choose to continue behaving suboptimally.

Proof. From Proposition 1, we know that the investor loses at most \(K\) from investing suboptimally for a small portion of the time. If we set costs to \(K\), it follows that the investor is better off using the suboptimal strategy.

Proposition 2. Consider an investor that is an expected utility maximizer with a neoclassical strict monotone increasing utility function \(u(d)\), a convex budget constraint, and an environment of regime shifts in extreme events. Denote the optimal risky asset demands during known typical and extreme regimes as \(d^T\) and \(d^E\), respectively, where \(d^E < d^T\). Denote the optimal risky demand when the investor does not know which regime obtains as \(\hat{d}\). In this setting, \(\hat{d}\) is bounded by \(d^T\) and \(d^E\), that is, \(d^E \leq \hat{d} \leq d^T\).

Proof. We shall proceed by contradiction. First, the hypothesis of strict monotone increasing utility function \(u(d)\) implies that for any feasible demands \(d^1\) and \(d^2\), the following relation holds:

\[
u(d^1) > u(d^2) \quad \text{if and only if} \quad d^1 > d^2.
\]

Second, the hypothesis of expected utility implies that if the investor faces a lottery between two alternative utilities in typical (T) and extreme (E) regimes \(u^T(d)\) and \(u^E(d)\) with probabilities \(1 - \alpha\) and \(\alpha\), then the expected value of her utility is given by

\[
EU = (1 - \alpha)u^T(d) + \alpha u^E(d).
\]

As an expected utility maximizer, the investor will choose \(\hat{d}\) in the feasible set \(D\) to maximize \(EU\), that is

\[
\hat{d} = \arg \max_{d \in D} (1 - \alpha)u^T(d) + \alpha u^E(d),
\]

for all \(\alpha \in [0, 1]\).
We now ascertain the value of $\hat{d}$ and proceed by contradiction. There are two cases to consider.

**Case 1:** Suppose that $\hat{d} > d^T$. Then it follows from (13) that $u^T(\hat{d}) > u^T(d^T)$. Now set $\alpha = 0$, which implies from (14) that $\hat{d} = \arg\max_{d \in D} u^T(d)$. But since $\hat{d}$ is in the feasible set, this means that $\hat{d}$ should have been chosen in the typical regime, as it gives higher utility. This contradicts that $d^T$ is optimal for the typical regime.

**Case 2:** Suppose that $\hat{d} < d^E$. Then it follows from (13) that $u^E(\hat{d}) < u^E(d^E)$. Now set $\alpha = 1$, which implies from (14) that $\hat{d} = \arg\max_{d \in D} u^E(d)$. But since $d^E$ is feasible, $d^E$ should have been chosen in (14), as it gives higher utility. This contradicts that $\hat{d}$ maximizes expected utility. Thus, in both cases we obtain a contradiction. Therefore $d^E \leq \hat{d} \leq d^T$, as was to be shown.

### A.1 Preliminary Results for Proposition 3

Recall from equation (3) in the text that the expression for the objective function is

$$U(W(d | \sigma^2)) = d\hat{\mu} + \frac{1}{2}d(1-d)\sigma^2 + \frac{1}{2}(1-\gamma)d^2\sigma^2 = d\hat{\mu} + \frac{\sigma^2}{2}[d - \gamma d^2]$$

$$= d \left( \frac{2\hat{\mu} + \sigma^2}{2} \right) - d^2 \left( \frac{\gamma \sigma^2}{2} \right)$$

(15)

Since investment is not necessarily optimal for the particular volatility regime, we clarify the payoff *conditional* on the prevailing regime, denoted $U[d|\sigma^2]$ or $U[d|m^2\sigma^2]$.

In order to simplify the proof of Proposition 3, we first calculate two differentials, namely $d^E - d^T$, and $(d^E)^2 - (d^T)^2$. To perform these computations, recall that the optimal demand vector from equation (6) is

$$d^L = d^T = \hat{\mu} + \frac{\sigma^2}{2\gamma \sigma^2}, \text{ with probability } 1 - \alpha$$

$$d^E = \hat{\mu} + \frac{(m\sigma)^2}{\gamma(m\sigma)^2} = \frac{2\hat{\mu} + m^2\sigma^2}{2\gamma m^2\sigma^2}, \text{ with probability } \alpha.$$  

(16)

**First Term** $d^E - d^T$: From expression (16), this is seen to be

$$d^E - d^T = \frac{2\hat{\mu} + m^2\sigma^2}{2\gamma m^2\sigma^2} - \frac{2\hat{\mu} + \sigma^2}{2\gamma \sigma^2}$$

$$= \frac{2\hat{\mu} + m^2\sigma^2 - 2\hat{\mu}m^2 - m^2\sigma^2}{2\gamma m^2\sigma^2} = \frac{2\hat{\mu} - 2\hat{\mu}m^2}{2\gamma m^2\sigma^2}$$

$$= \frac{\hat{\mu}[1 - m^2]}{\gamma m^2\sigma^2}.$$  

(17)
Second Term \((d^T)^2 - (d^E)^2\): From expression (16), we see that
\[
(d^E)^2 = \frac{(2\hat{\mu} + m^2\sigma^2)^2}{4\gamma^2m^4\sigma^4} = \frac{4\hat{\mu}^2 + 4\hat{\mu}m^2\sigma^2 + m^4\sigma^4}{4\gamma^2m^4\sigma^4}
\]
and
\[
(d^T)^2 = \frac{(2\hat{\mu} + \sigma^2)^2}{4\gamma^2\sigma^4} = \frac{4\hat{\mu}^2 + 4\hat{\mu}\sigma^2 + \sigma^4}{4\gamma^2\sigma^4}.
\]

Therefore the differential is
\[
(d^T)^2 - (d^E)^2 = \frac{4\hat{\mu}^2 + 4\hat{\mu}\sigma^2 + \sigma^4}{4\gamma^2\sigma^4} - \frac{4\hat{\mu}^2 + 4\hat{\mu}m^2\sigma^2 + m^4\sigma^4}{4\gamma^2m^4\sigma^4}
\]
\[
= \frac{m^4[4\hat{\mu}^2 + 4\hat{\mu}\sigma^2 + \sigma^4] - 4\hat{\mu}^2 - 4\hat{\mu}m^2\sigma^2 - m^4\sigma^4}{4\gamma^2m^4\sigma^4}
\]
\[
= \frac{4\hat{\mu}^2m^4 + 4\hat{\mu}m^4\sigma^2 - 4\hat{\mu}^2 - 4\hat{\mu}m^2\sigma^2}{4\gamma^2m^4\sigma^4}
\]
\[
= \frac{\hat{\mu}^2m^4 + \hat{\mu}m^4\sigma^2 - \hat{\mu}^2 - \hat{\mu}m^2\sigma^2}{\gamma^2m^4\sigma^4}
\]
\[
= \frac{\hat{\mu}^2[m^4 - 1] + \hat{\mu}\sigma^2[m^4 - m^2]}{\gamma^2m^4\sigma^4}.
\]

Proposition 3. The cost to investors of suboptimal behavior during extremes is bounded above by a constant \(K\), which is proportional to squared, standardized excess returns \((\frac{\hat{\mu}}{\sigma})^2\).

Proof. We need to show that the utility loss \(\Delta EU\) from investing a proportion \(d^T\) instead of \(d^E\) during extreme periods is of the form \(\Delta EU \leq K\), where \(K = \theta \left(\frac{\hat{\mu}}{\sigma}\right)^2\) for some positive, finite \(\theta\). In order to compute the utility loss, we just calculate the investor’s objective function (15) in both cases.

Optimal: The optimal strategy is to invest \(d^E\), which yields payoff \(U[W(d^E|m^2\sigma^2)]\), i.e.
\[
U[W(d^E|m^2\sigma^2)] = d^E \left(\frac{2\hat{\mu} + m^2\sigma^2}{2}\right) - (d^E)^2 \left(\frac{\gamma m^2\sigma^2}{2}\right).
\]

Suboptimal: In similar fashion, the suboptimal payoff \(U[W(d^T|m^2\sigma^2)]\) can be calculated as
\[
U[W(d^T|m^2\sigma^2)] = d^T \left(\frac{2\hat{\mu} + m^2\sigma^2}{2}\right) - (d^T)^2 \left(\frac{\gamma m^2\sigma^2}{2}\right).
\]

Thus the payoff differential \(\Delta EU \equiv U[W(d^E|m^2\sigma^2)] - U[W(d^T|m^2\sigma^2)]\) is given by
\[
\Delta EU = (d^E - d^T) \left[\frac{2\hat{\mu} + m^2\sigma^2}{2}\right] + \frac{\gamma m^2\sigma^2}{2} \left[(d^T)^2 - (d^E)^2\right].
\]

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Now substitute the expressions for \((d^E - d^T)\) and \((d^T)^2 - (d^E)^2\) from (17) and (18) to obtain

\[
\Delta EU = \frac{\mu - \mu m^2}{\gamma m^2 \sigma^2} \left[ \frac{2 \mu + m^2 \sigma^2}{2} \right] + \frac{\gamma m^2 \sigma^2}{2} \left[ \frac{\mu^2 m^4 + \mu m^4 \sigma^2 - \mu^2 - \mu m^2 \sigma^2}{\gamma^2 m^4 \sigma^4} \right]
\]

\[
= \frac{(\mu - \mu m^2)(2\mu + m^2 \sigma^2)}{2\gamma m^2 \sigma^2} + \frac{\mu^2 m^4 + \mu m^4 \sigma^2 - \mu^2 - \mu m^2 \sigma^2}{2\gamma m^2 \sigma^2}
\]

\[
= \frac{2\mu^2 + \mu m^2 \sigma^2 - 2\mu^2 m^2 - \mu m^4 \sigma^2 + \mu^2 m^4 + \mu m^4 \sigma^2 - \mu^2 - \mu m^2 \sigma^2}{2\gamma m^2 \sigma^2}
\]

\[
= \frac{\mu^2 - 2\mu^2 m^2 + \mu^2 m^4}{2\gamma m^2 \sigma^2} = \frac{\mu^2 [1 - 2m^2 + m^4]}{2\gamma m^2 \sigma^2}
\]

\[
= \frac{\mu^2 [1 - m^2]^2}{2\gamma m^2 \sigma^2}
\]

The expression in (19) is of the form \(K = \theta \left( \frac{\mu}{\sigma} \right)^2\), where \(\theta = \frac{1 - m^2 \sigma^2}{2\gamma m^2}\), as was to be shown.

\[
\square
\]

**Corollary 2.** If the costs of learning about extreme events are above a threshold, the investor will prefer to over-invest during extreme periods.

**Proof.** From the previous proposition, it follows that if costs are above \(K\), the investor will be better off by over-investing. \(\square\)

### A.2 Preliminary Results for Propositions 4 and 5

There are seven quantities that we calculate for use in proving Propositions 4 and 5. These terms are all computed using the payoff expression in (15), and the \(d^L\) and \(d^R\) terms in (16).

**TERM 1.** The first term, \(EU[d^L|\sigma^2]\), is given by

\[
EU[d^L|\sigma^2] = \left[ \frac{2\mu + \sigma^2}{2\gamma \sigma^2} \right] \left[ \frac{2\mu + \sigma^2}{2} \right] - \left( \frac{2\mu + \sigma^2}{4\gamma^2 \sigma^4} \right) \left[ \frac{\gamma \sigma^2}{2} \right]
\]

\[
= \frac{(2\mu + \sigma^2)^2}{4\gamma \sigma^2} - \frac{(2\mu + \sigma^2)^2}{8\gamma \sigma^2}
\]

\[
= \frac{(2\mu + \sigma^2)^2}{8\gamma \sigma^2}
\]

\[
(20)
\]

**TERM 2.** The second term, \(EU[d^P|\sigma^2]\), is

\[
EU[d^P|\sigma^2] = \frac{2\mu + \sigma^2 - \gamma \sigma^2}{2} = \frac{2\mu - \sigma^2(\gamma - 1)}{2}
\]

\[
(21)
\]

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TERM 3. The third term, \( EU[d^E|m^2\sigma^2] \), is

\[
EU[d^E|m^2\sigma^2] = \frac{2\mu + m^2\sigma^2}{2\gamma m^2\sigma^2} \left[ \frac{2\mu + m^2\sigma^2}{2} - \frac{(2\mu + m^2\sigma^2)^2}{4\gamma m^4\sigma^4} \right] \left[ \frac{\gamma m^2\sigma^2}{2} \right] = \frac{(2\mu + m^2\sigma^2)^2}{4\gamma m^2\sigma^2} - \frac{(2\mu + m^2\sigma^2)^2}{8\gamma m^2\sigma^2} = \frac{(2\mu + m^2\sigma^2)^2}{8\gamma m^2\sigma^2}.
\]

TERM 4. The fourth term is

\[
EU[r_b(d^L - 1)|m^2\sigma^2] = \frac{r_b[2\mu + \sigma^2 - 2\gamma\sigma^2]}{2\gamma\sigma^2} \left[ \frac{2\mu + \sigma^2}{2} - \frac{r_b^2(2\mu + \sigma^2 - 2\gamma\sigma^2)^2}{4\gamma^2\sigma^4} \right] \left[ \frac{\gamma\sigma^2}{2} \right] = \frac{r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + \sigma^2)}{4\gamma\sigma^2} \left[ \frac{r_b^2(2\mu - \sigma^2(2\gamma - 1))^2}{8\gamma\sigma^2} \right] = \frac{2r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + \sigma^2) - r_b^2[2\mu - \sigma^2(2\gamma - 1)]^2}{8\gamma\sigma^2}.
\]

TERM 5. The fifth term is

\[
EU[r_b(d^L - 1)|\sigma^2] = \frac{r_b[2\mu + \sigma^2 - 2\gamma\sigma^2]}{2\gamma\sigma^2} \left[ \frac{2\mu + \sigma^2}{2} - \frac{r_b^2(2\mu + \sigma^2 - 2\gamma\sigma^2)^2}{4\gamma^2\sigma^4} \right] \left[ \frac{\gamma\sigma^2}{2} \right] = \frac{r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + \sigma^2)}{4\gamma\sigma^2} \left[ \frac{r_b^2(2\mu - \sigma^2(2\gamma - 1))^2}{8\gamma\sigma^2} \right] = \frac{2r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + \sigma^2) - r_b^2[2\mu - \sigma^2(2\gamma - 1)]^2}{8\gamma\sigma^2}.
\]

TERM 6. The sixth term, \( caEU[r_b(d^L - 1)|m^2\sigma^2] + (1 - ca)EU[r_b(d^L - 1)|\sigma^2] \), combines (23) and (24) to yield

\[
caEU[r_b(d^L - 1)|m^2\sigma^2] + (1 - ca)EU[r_b(d^L - 1)|\sigma^2]
= \frac{ca \left[ 2r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + m^2\sigma^2) - m^2r_b^2[2\mu - \sigma^2(2\gamma - 1)]^2 \right]}{8\gamma\sigma^2} + (1 - ca) \frac{[ca(2\mu + m^2\sigma^2) + (1 - ca)(2\mu + \sigma^2)]}{8\gamma\sigma^2} = \frac{2r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + \sigma^2) - r_b^2[2\mu - \sigma^2(2\gamma - 1)]^2}{8\gamma\sigma^2} + \frac{2r_b[2\mu - \sigma^2(2\gamma - 1)](2\mu + m^2\sigma^2) - r_b^2[2\mu - \sigma^2(2\gamma - 1)]^2}{8\gamma\sigma^2}.
\]
TERM 7. The seventh term, $EU[d^P|m^2\sigma^2]$, is used only in the proof of Proposition 5, and is given by

$$EU[d^P|m^2\sigma^2] = \frac{2\hat{\mu} + m^2\sigma^2 - \gamma m^2\sigma^2}{2} = \frac{2\hat{\mu} - m^2\sigma^2(\gamma - 1)}{2}. \quad (26)$$

Proposition 4. The expected net benefit of leverage ($d > 1$) in period 1 is bounded, for an investor who knows that the environment features regime-switching in extreme events. This effect depends on the values of $\mu$ and $\sigma^2$.

Proof. We need to show that the difference in the investor’s objective function from leveraged investment, $P^L$, minus that from prudent investment, $P^P$, is bounded. That is, we must show that $|P^L - P^P| < \infty$. It suffices to show that the difference comprises a sum or product of bounded finite terms. In order to do this, we calculate the investor’s expected payoff from choosing prudent and leveraged investment levels. We denote $EU$ as the expected utility, from the objective function in equation (15). We assume that period 1 is a typical regime with variance $\sigma^2$. In period 1 the investor decides whether to borrow and invest $d^E$, or else invest the prudent amount $d^P = 1$. In period 2 the investor will choose optimally for that period: either $d^E$ if it is extreme, or else $d^P$ for typical economic climates. First we compute the payoff $P^P$ as follows:

$$P^P = EU[d^P|\sigma^2] + \beta(\alpha EU(d^E|m^2\sigma^2) + (1 - \alpha) EU(d^P|\sigma^2)), \quad (27)$$

Then we compute the payoff from leverage, $P^L$, as follows. In this case, the investor has to repay borrowing $r_b(d^L - 1)W_0$ in the second period, where we normalize $W_0 = 1$ to obtain $r_b(d^L - 1)$. Hence the payoff is

$$P^L = EU[d^L|\sigma^2] + \beta c\alpha (EU[d^E|m^2\sigma^2] - EU[r_b(d^L - 1)|m^2\sigma^2]) \quad (28)$$

Now to calculate the expected net costs for leverage we compute $P^L - P^P$ from (27) and (28):

$$P^L - P^P = EU[d^L|\sigma^2] + EU[d^P|\sigma^2][\beta(1 - \alpha\alpha) - 1 - \beta(1 - \alpha)]$$
$$+ \beta\{\alpha EU(d^E|m^2\sigma^2) - \alpha EU(d^P|m^2\sigma^2)\}$$
$$+ \beta\{-\alpha EU[r_b(d^L - 1)|m^2\sigma^2] - (1 - \alpha\alpha) EU[r_b(d^L - 1)|\sigma^2]\}$$
$$= EU[d^L|\sigma^2] - \{1 + \alpha\beta(c - 1)|EU[d^E|\sigma^2] + \alpha\beta(c - 1)EU[d^P|m^2\sigma^2]$$
$$\alpha\beta(c - 1)EU[r_b(d^L - 1)|m^2\sigma^2] + (1 - \alpha\alpha) EU[r_b(d^L - 1)|\sigma^2]\}).$$

The expression in (29) contains terms from our above Preliminary Results, in equations (20) to (25). Substituting these terms into (29) yields

$$P^L - P^P = \frac{(2\hat{\mu} + \sigma^2)^2}{8\gamma\sigma^2} - [1 + \alpha\beta(c - 1)] \left(\frac{2\hat{\mu} - \sigma^2(\gamma - 1)}{2}\right) + \alpha\beta(c - 1) \left(\frac{(2\hat{\mu} + m^2\sigma^2)^2}{8\gamma m^2\sigma^2}\right)$$
$$- \beta \left(\frac{2r_b[2\hat{\mu} - \sigma^2(\gamma - 1)][\alpha\gamma\sigma^2(m^2 - 1) + 2\hat{\mu} + \sigma^2]}{8\gamma^2}\right)$$
$$+ \beta \left(\frac{r_b^2[2\hat{\mu} - \sigma^2(\gamma - 1)]^2[1 + \alpha\gamma(m^2 - 1)]}{8\gamma^2}\right). \quad (30)$$

\[\text{In the second period agents cannot borrow to invest } d^E, \text{ since it is the end of economic activity. Therefore if it is a typical period, they just invest as much as they can, } d^P = 1.\]
Under the maintained assumptions, the expression in (30) comprises only sums and products of finite bounded terms in \( \mu \) and \( \sigma^2 \). Therefore it is a bounded function of \( \mu \) and \( \sigma^2 \), as was to be shown. The final expression for \( P^L - P^P \) in (30) is calibrated to US data in Figure 3.

\[ \square \]

**Proposition 5.** The expected net loss from suboptimal investment in period 2 is bounded, for an investor who does not understand that the environment features regime-switching in extreme events.

**Proof.** We have to show that \( \Delta EU \leq K \), for some positive constant \( K \). The investor assumes the world is always typical, and invests \( d^L > 1 \) in period 1 and the maximal \( d^P = 1 \) in period 2. The associated payoff is denoted \( \hat{P}^L \). To calculate the net expected cost we compute the difference between payoff to the optimal strategy \( P^L \) and its suboptimal counterpart \( \hat{P}^L \). That is, we compute \( P^L - \hat{P}^L \). Below we first present the optimal, then suboptimal payoffs.

**Optimal Payoff** \( P^L \). This is the same as above in (28):

\[
P^L = EU[d^L|\sigma^2] + \beta c \alpha \left( EU[d^E|m^2\sigma^2] - EU[r_b(d^L - 1)|m^2\sigma^2] \right) + \beta (1 - c \alpha) \left( EU[d^P|\sigma^2] - EU[r_b(d^L - 1)|\sigma^2] \right).
\]  

(31)

**Suboptimal Payoff** \( \hat{P}^L \). The strategy here involves investing \( d^L \) in period 1. Then in period 2 the investor repays any borrowing, and since she mistakenly believes the world is always in the typical regime, she always demands the most she can, \( d^P = 1 \), regardless of whether the realized regime is typical or extreme. To compute the results, we proceed as follows. If she over-invests by choosing \( d^L \) in the first period, the likelihood of extremes raises from \( \alpha \) to \( c \alpha \), and her payoff \( \hat{P}^L \) is

\[
\hat{P}^L = EU[d^L|\sigma^2] + \beta c \alpha \left( EU[d^P|m^2\sigma^2] - EU[r_b(d^L - 1)|m^2\sigma^2] \right) + \beta (1 - c \alpha) \left( EU[d^P|\sigma^2] - EU[r_b(d^L - 1)|\sigma^2] \right).
\]

(32)

**Utility Differential** \( P^L - \hat{P}^L \). Using equations (31) and (32), we obtain

\[
P^L - \hat{P}^L = \beta c \alpha \{ EU[d^E|m^2\sigma^2] - EU[d^P|m^2\sigma^2] \}.
\]

This can be completed using Terms 3 and 7 in equations (22) and (26) above:

\[
P^L - \hat{P}^L = \beta c \alpha \left[ \frac{(2\mu + m^2\sigma^2)^2}{8\gamma m^2\sigma^2} - \frac{2\mu - m^2\sigma^2(\gamma - 1)}{2} \right]
\]  

(33)

The expression in (33) represents the net utility gain from following the optimal versus the suboptimal strategy in the case of leverage, or equivalently, the net loss from the suboptimal strategy. Upon inspection this quantity can be confirmed as bounded.

\[ \square \]

**Proposition 6.** Suppose a risk-averse investor faces extreme events as described in (10), and currently invests \( d^* = 0 \) in the risky asset, receiving finite returns and wealth \( W^* \). If she deviates from \( d^* \) by choosing an alternative investment strategy \( \hat{d} > d^* \) even a small proportion \( \theta \) of the time, her expected utility loss is potentially unbounded. Conversely, a risk-loving investor’s expected gain is potentially unbounded.
Proof. Intuitively, if a risk-averse (risk-loving) investor deviates from the status quo and extreme events occur, the utility function is unboundedly negative (positive), so her expected utility loss (gain) is unbounded. More formally, we must show that for a risk-averse investor the expected utility loss is $\Delta EU = \infty$. First, denote the alternative wealth level from deviating to invest $\hat{d}$ as $\hat{W}(\hat{d})$. Now observe that expected utility loss is the difference between original utility and the alternative utility that results from changing her strategy with probability $\theta$. Thus $\Delta EU \equiv U(W^*) - [\theta U(\hat{W}) + (1 - \theta)U(W^*)]$, where we drop the argument in $W(\cdot)$ for simplicity. Computing the expected utility loss, we obtain

$$
\Delta EU \equiv U(W^*) - [\theta U(\hat{W}) + (1 - \theta)U(W^*)] 
\quad = \theta[U(W^*) - U(\hat{W})].
$$

By risk aversion, the investor’s utility decreases with variance, and $U(\hat{W}) = -\infty$. Therefore the quantity in (34) is unbounded above, as was to be shown. A similar argument applies to the risk loving investor.

Corollary 3. Suppose there is a small likelihood $\theta > 0$ of extreme events, and that investor preferences respond to risk, i.e. the investor is either risk-averse or risk-loving. If the investor can choose between investing immediately or first learning about the likelihood of extremes, she will prefer to incur any finite costs to learning about extreme events, before investing in the risky asset.

Proof. From Proposition 6, we know that a risk averse investor faces unbounded expected utility losses by investing in the risky asset. Therefore she will prefer to pay any finite costs $c < \infty$ before investing in the risky asset. The risk loving investor faces unbounded gains in expected utility terms, and will therefore also wish to learn about the extreme events, otherwise she might lose an opportunity to invest in an asset that yields infinite expected utility.

Proposition 7. Consider an investor with current expected wealth $W^* < \infty$ who decides how much to invest in the risky asset, and who is Aware that there may be an extreme regime as in (10). The investor does not know the true likelihood of extremes $\alpha^*$, and therefore has to estimate it. Even if her estimate is biased, as long as it is positive, her optimal risky demand will be the same as if she knew the true likelihood of extremes with certainty.

Proof. Denote the investor’s estimate of the likelihood of extremes as $\hat{\alpha} > 0$, such that $\hat{\alpha} \neq \alpha^*$. We consider the optimal demands for preferences that are risk averse, risk-loving, and risk-neutral, in turn. In each case we compute the change in expected utility $\Delta EU$ as in (34) above. First, for a risk-averse investor the expected utility loss from positive investment in the risky asset based on $\hat{\alpha}$ is $\Delta EU = \hat{\alpha}[U(W^*) - U(\hat{W})]$, similar to (34) above. Because the $U(\hat{W})$ term equals negative infinity, $\Delta EU$ is infinitely large. Therefore a risk-averse investor would invest $d = 0$, just as if she knew the correct $\alpha^*$. Second, a risk-loving investor will have infinitely large expected utility Gains and invest as much as possible just as if she knew the correct $\alpha^*$. Finally, a risk-neutral investor experiences no utility change, $\Delta EU = 0$, because her utility function is unaffected by wealth variance. Consequently a risk-neutral investor will also choose the same investment as if she knew the true likelihood of extremes.
**Corollary 4.** Suppose an investor is open-minded that there is a positive likelihood of extreme events, and chooses a positive, potentially biased, estimate of extreme likelihood, \( \hat{\alpha} > 0 \). Then only if she is risk-neutral or risk-loving will she will invest in the risky asset.

**Proof.** From Propositions 6 and 7, as long as there is positive estimate of extreme likelihood, a risk averse investor will stay out of the market, while risk-neutral and risk-loving investors will invest in the risky asset. \( \square \)

**Corollary 5.** Suppose an investor currently invests in the risky asset, i.e. \( d > 0 \). If she is Unaware that the extreme regime exists and has risk-averse (risk-loving) preferences, she will subsequently incur infinite utility losses (gains), while a risk-neutral investor will face zero utility losses.

**Proof.** Since the investor believes there is only a typical regime with finite variance, i.e. \( \alpha = 0 \) in equation (10), she continues her existing investment strategies \( d > 0 \), based on the typical regime. Since a risk-averse (risk-loving) investor utility decreases (increases) monotonically with wealth variance, she will suffer infinite utility loss (gain) when the extreme regime occurs. Conversely, a risk-neutral investor’s utility does not respond to wealth variance, so her utility is unaffected in the extreme regime. \( \square \)